Lateral-Torsional Buckling Modification Factors in Steel I-Shaped Members: Recommendations Using Energy-Based Formulations

NAMITA NAYAK, P.M. ANILKUMAR, and LAKSHMI SUBRAMANIAN

ABSTRACT

Lateral torsional buckling (LTB) is of concern in long-span flexural members, particularly in the negative flexure regions of continuous-span, steel I-shaped members and during construction. While the elastic critical LTB capacity of a simply supported I-shaped member subjected to uniform moment has a closed-form solution, most LTB modification factors for beams subjected to moment gradients in the literature are empirical and work well only for specific loading and boundary conditions. This paper investigates the suitability of the different LTB modification factors in literature and design specifications for various loading and boundary conditions, accomplished via comparisons with analytical solutions using the Rayleigh-Ritz method and numerical solutions from finite element analyses. The analytical LTB modification factors are derived for doubly symmetric I-shaped members with different combinations of ideal flexural and torsional boundary conditions (simply supported and fixed) and subjected to different loading scenarios. The validity of the LTB modification factors determined using the Rayleigh-Ritz method and other formulae in the literature are also assessed for realistic intermediate restraint conditions, which are neither fully pinned nor fixed, by examining laterally continuous beams. Demonstrating that current design specifications for elastic critical LTB modifications are overly conservative for beams with complete or partial warping fixity, the authors recommend practical and simple alternatives to design such beams.

Keywords: lateral torsional buckling, LTB modification factor, Rayleigh-Ritz method, warping restraints, continuous beams.

INTRODUCTION

This paper investigates the elastic critical lateral torsional buckling (LTB) capacities of doubly symmetric, steel I-shaped members loaded at their centroidal axes, considering a spectrum of loading and boundary conditions. The classical solution for the elastic critical lateral torsional buckling capacity was first proposed by Timoshenko (1936) for simply supported, doubly symmetric, I-shaped members subjected to uniform moment, which is used worldwide as the elastic critical lateral torsional buckling capacity. Several empirical formulae for LTB modification factors have since been developed to account for the enhancement in the flexural capacities of beams with nonuniform moments within the unbraced spans (such as Salvadori, 1956; Nethercot and Rockey, 1972; Kirby and Nethercot, 1979; Serna et al., 2006; Wong and Driver, 2010). Design codes and specifications (BS 5950-1, 2000; AASHTO, 2020; AISC, 2022) employ these modification factors with or without elastic lateral effective length factors. The British standard recommends using an effective length based on the restraint conditions at the ends of the unbraced segments. While AISC *Specification* (2022) Equation F2-4 stipulates the use of a full unbraced length instead of an effective unbraced length by defining the unbraced length as the distance between lateral braces, its Commentary discusses using an effective length factor based on the end restraints as per Ziemian (2010). Similarly, the AASHTO *Specification* Commentary (2020) explores using an effective length factor in rehabilitation design or extraordinary circumstances, which may be calculated based on Nethercot and Trahair (1976) and Ziemian (2010). The recommendation of $K = 1.0$ in the American specifications (AASHTO, 2020; AISC, 2022) can lead to significantly conservative estimates of elastic critical LTB capacities of I-shaped members, whose ends are torsionally fixed (both twist and warping are fixed). This is true irrespective of the in-plane flexural boundary conditions. Such beams are practically found in cases such as rigid beam-column joints or in typical moment connection details. However, using an elastic effective length factor, K (as suggested in the AISC and AASHTO *Specification* Commentaries), along with the current equations for the LTB modification factors (C_b) may lead to inaccurate estimates of the flexural capacities of I-shaped members.

Namita Nayak, Research scholar, Department of Civil Engineering, Indian Institute of Technology Madras, Tamil Nadu, India. Email: ce20d041@smail.iitm.ac.in

P.M. Anilkumar, Institute of Structural Analysis, Gottfried Wilhelm Leibniz University Hannover, Hannover, Germany. Email: a.nair@isd.uni-hannover.de

Lakshmi Subramanian, Assistant Professor, Department of Civil Engineering, Indian Institute of Technology Madras, Tamil Nadu, India. Email: lakshmipriya@iitm.ac.in (corresponding)

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An inaccurate estimation of C_b, and thereby, the elastic critical buckling capacities, affects the entire beam design curve. For example, the AISC *Specification* beam design curve consists of three parts: the plateau for short unbraced lengths ($L_b < L_p$) with the maximum cross-section capacity, an elastic LTB curve for long unbraced lengths ($L_b > L_r$), and an inelastic LTB equation that is a linear interpolation between the maximum cross-section capacity and the elastic LTB curves. The inelastic and elastic LTB design capacities in the AISC and AASHTO *Specifications* are scaled by C_b for nonuniform moment loading conditions, limited by the maximum cross-section capacity. Such scaling often leads to an extended plateau length several times larger than L_p, and a greatly enhanced inelastic LTB capacity. Hence, an inaccurate estimation of C_b may result in either overly conservative or unconservative estimates of the beam capacities in a significant portion of the beam design space. Subramanian and White (2017a) also noted that the extended plateau resulting from scaling the flexural capacity by C_b in the inelastic LTB region tends to overestimate the true strengths even for beams which are free to warp at their ends ($K = 1.0$). Although the available literature provides numerous formulations for estimating C_b, the existing equations are generally fit to the results of numerical parametric studies for specific loading and boundary conditions.

The objective of this study is to evaluate the appropriateness of the existing C_b factors in design specifications and literature for ideal boundary conditions, including fork boundary conditions (flexurally and torsionally simply supported), fully fixed (flexurally and torsionally) conditions, and flexurally simply supported and torsionally fixed boundary conditions. More practical conditions, including laterally continuous beams, are also subsequently examined. The loading conditions include linear moment gradients, a concentrated load at mid-span, and uniformly distributed loads. The available empirical equations are compared with analytical solutions using the Rayleigh-Ritz method and finite element (FE) simulations. The Rayleigh-Ritz method is an energy-based approach, wherein the LTB capacity is obtained by minimizing the total potential energy of the system. Although the energy method has been previously used by others (Timoshenko, 1936; Galambos and Surovek, 2008; Yoo and Lee, 2011) to calculate the elastic critical buckling moments, those studies were limited to simply supported I-shaped members and cantilever beams subjected to concentrated loads. This work seeks to establish accurate formulations for predicting the elastic critical moment of steel I-shaped members for standard loading and end-restraint conditions, while also defining the specific conditions for which the available commonly used equations are most suitable. The results from the Rayleigh-Ritz method are also compared with FE test simulations. The comparisons with the literature and FE simulations also help identify practical design scenarios where one must exercise caution when using the existing equations in the design specifications and the recommendations in the commentaries for LTB resistances.

Following the studies on beams with ideal boundary conditions, this paper looks at practical design conditions with laterally continuous beams. In beams with intermediate lateral braces, the critical lateral spans are restrained by their adjoining segments. A correct estimate of the flexural capacity of such beams depends on the effective length of the critical unbraced span and the appropriate C_b factor. While the effective lateral length factor, K, for the ideal boundary conditions, may be taken as 1.0 for torsionally simply supported conditions, and 0.5 for torsionally fixed conditions, the appropriate K for laterally continuous beams is determined using other methods.

White (2008) and White and Jung (2008) briefly described the evolution of the AISC *Specification* beam design equations, which are largely a fit to a vast body of experimental data. They explained that the current coefficient in the equation for L_p (AISC *Specification* Equation F4-7) may be taken as devoid of any implicit effective length factors. They further cautioned that employing $K = 1.0$, as outlined in the AISC *Specification*, is conservative for several design conditions and recommended using the effective length proposed by Nethercot and Trahair (1976) and Galambos (1998). Nethercot and Trahair first proposed a method to estimate the elastic effective length factor for laterally continuous beams, akin to the method for braced columns. They stipulated that the restraints to the critical span from the immediately adjoining segments are functions of the loading in both the critical and restraining segments, and the far-end boundary conditions. Later, Subramanian et al. (2018) discussed the effect of inelasticity in the critical segments, leading to a consideration of $K_{inelastic}$ in interpreting experimental test data. They demonstrated an improved reliability when the plateau length, L_p, was reduced to a coefficient of 0.63 instead of 1.1, and the anchor point for the elastic stresses, F_L, was decreased to $0.5F_{yc}$, as recommended in Kim (2010) and Subramanian and White (2017b).

More recently, John and Subramanian (2019) proposed modifications to the original method by Nethercot and Trahair (1976), noting that the restraint also depends on whether the adjoining segment braces the critical span at the location of the maximum or the minimum moment within the critical span. Additionally, John and Subramanian discussed situations where the farther segments further influence the restraints from the spans immediately adjoining the critical span. They also observed conditions where the critical lateral span may be identified incorrectly. This paper examines the various C_b formulations for laterally continuous beams by applying the effective length factors from both the Nethercot and Trahair (1976) method and the modified methods suggested by John and Subramanian (2019).

These exhaustive comparisons of the LTB modification factors for ideal boundary conditions and laterally continuous beams with partial restraint conditions lead the authors to recommend design methods better suited for a broader range of beam design conditions.

EVALUATION OF M_{cr} USING EXISTING DESIGN CODES AND EMPIRICAL FORMULAE

Timoshenko (1936) derived the elastic critical LTB capacity for flexurally and torsionally simply supported, doubly symmetric, I-shaped members subjected to a uniform moment. The AISC *Specification* Commentary discusses using an elastic effective length instead of the full unbraced length to enhance the elastic critical buckling moment by using the method prescribed in Ziemian (2010), which considers the lateral and torsional boundary conditions in the beams. This modified elastic critical LTB capacity, M_{ocr}, may be written as given in Equation 1 by considering the effective length KL for the different flexural and torsional boundary conditions. This basic critical moment equation is typically modified for different loading and boundary conditions by multiplying the expression for M_{ocr} with the LTB modification factor, C_b. Some of the commonly used expressions for C_b are listed in Table 1.

$$M_{ocr} = \sqrt{\frac{\pi^2 EI_y}{(KL)^2}\left(\frac{\pi^2 EI_w}{(KL)^2} + GJ\right)} \tag{1}$$

where

E = Young's modulus of elasticity, ksi

G = elastic shear modulus, ksi

I_y = minor axis moment of inertia, in.^{4}

I_w = warping constant, in.^{4}

J = St.-Venant torsional constant, in.^{4}

L = lateral unbraced length, in.

The American specifications use Equation 2 (AISC *Specification* Equation F2-4 and AASHTO *Specification* Equation A6.3.3) to estimate the elastic critical LTB stress of an I-shaped member. This equation is similar to Timoshenko's solution for a flexurally and torsionally simply supported, doubly symmetric, I-shaped member (with K = 1.0) subjected to uniform moment.

$$F_{cr} = \frac{C_b \pi^2 E}{\left(\frac{L_b}{r_{ts}}\right)^2}\sqrt{1 + 0.078\frac{Jc}{S_x h_o}\left(\frac{L_b}{r_{ts}}\right)^2} \tag{2}$$

$$r_{ts}^2 = \frac{\sqrt{I_y C_w}}{S_x}$$

where

C_b = LTB modification factor for nonuniform moment diagrams

L_b = lateral unbraced length, in.

S_x = elastic section modulus taken about the major axis of the cross section, in.^{3}

c = 1.0 for doubly symmetric I-sections

h_o = distance between the flange centroids, in.

Table 1 provides expressions for LTB modification factors from the literature, commonly derived from a fit to the data from finite element or finite difference methods for moment gradients. These equations are derived for specific loading conditions but are, however, usually applied to all loading and boundary conditions. For example, the equation proposed by Salvadori (1956), incorporated in the AASHTO *Specification*, applies to members with linear bending moment diagrams between the two braced points. While this equation is simple to use and yields good results for bending in single curvature, it is significantly conservative when applied to beams with double curvature. Similarly, Nethercot and Rockey (1972) provided the LTB modification factors for beams with warping-fixed boundary conditions subjected to a concentrated load at mid-span and a uniformly distributed load (Ziemian, 2010); however, the in-plane boundary conditions in the warping-fixed cases were not specified.

Serna et al. (2006) found that the equations provided in the AISC *Specification* and the British standard (BS 5950-1, 2000) are unconservative for warping-fixed conditions. They proposed an alternate equation, where the LTB modification factor is a function of the torsional boundary conditions. Wong and Driver (2010) opined that the equation proposed by Serna et al. (2006) overestimates the elastic critical LTB capacity, and they proposed the C_b factor presented in Table 1. The equation developed by Wong and Driver (2010) fits the numerical data considering the effect of moment gradients in beams whose ends are free to warp.

The British standard BS 5950-1 (2000) employs a LTB modification coefficient similar to the AISC *Specification* with different coefficients for the bending moments at quarter-, mid-, and three-quarter span locations. The British standard further recommends using an effective length factor, K, of 1.0 and 0.7 for warping-free and fixed boundary conditions, respectively. While the AISC *Specification* recommends using the C_b factor (Equation F1-1) proposed by Kirby and Nethercot (1979), several other C_b factors are presented in the Commentary. These equations include Wong and Driver's (2010) equation (AISC *Specification* Commentary Equation C-F1-2b) for nonlinear moment diagrams, and the C_b factor proposed by Yura and Helwig

Table 1. Review of LTB Modification Factors, C_b, from the Literature		
Source	**Equation**	**Remarks**
Salvadori (1956) AISC *Specification* (2022) Equation C-F1-1 AASHTO *Specification* (2020)* Equation A6.3.3.7	$C_b = 1.75 + 1.05\left(\frac{M_1}{M_2}\right) + 0.30\left(\frac{M_1}{M_2}\right)^2 \leq 2.30$	M_1—smaller moment at the end of the unbraced length M_2—larger moment at the end of the unbraced length The ratio of M_1 to M_2 is positive for double curvature and negative for single curvature bending
Nethercot and Rockey (1972) Ziemian (2010)	1.35[a] 1.13[b] $C_b = 1.92 - 0.42\left(\frac{\pi}{L}\sqrt{\frac{EI_w}{GJ}}\right)^2 + 1.85\left(\frac{\pi}{L}\sqrt{\frac{EI_w}{GJ}}\right)$[c] $C_b = 1.64 - 0.41\left(\frac{\pi}{L}\sqrt{\frac{EI_w}{GJ}}\right)^2 + 1.77\left(\frac{\pi}{L}\sqrt{\frac{EI_w}{GJ}}\right)$[d]	[a] Flexurally and torsionally simply supported with a concentrated load at mid-span [b] Flexurally and torsionally simply supported with a uniformly distributed load [c] Flexurally and torsionally fixed with a concentrated load at mid-span [d] Flexurally and torsionally fixed with a uniformly distributed load
Kirby and Nethercot (1979) (AISC *Specification* Equation F1-1)	$C_b = \frac{12.5M_{max}}{2.5M_{max} + 3.0M_A + 4.0M_B + 3.0M_C}$	M_{max}—absolute value of the maximum moment in the unbraced segment M_A, M_B, M_C—absolute values of the moments at the quarter-, mid-, and three-quarter points of the unbraced segment
British standard (BS 5950-1, 2000)	$C_b = \frac{M_{max}}{0.20M_{max} + 0.15M_2 + 0.50M_3 + 0.15M_4} \leq 2.27$	M_{max}—absolute value of the maximum moment in the unbraced segment M_2, M_3, M_4—absolute values of the moments at the quarter-, mid-, and three-quarter points of the unbraced segment
Serna et al. (2006)	$C_b = \frac{\sqrt{\sqrt{k}A_1 + \left[\frac{(1-\sqrt{k})}{2}A_2\right]^2} + \frac{(1-\sqrt{k})}{2}A_2}{A_1}$	$A_1 = \frac{M_{max}^2 + 9kM_2^2 + 16M_3^2 + 9kM_4^2}{(17+18k)M_{max}^2}$ $A_2 = \left\lvert\frac{M_{max} + 4M_1 + 8M_2 + 12M_3 + 8M_4 + 4M_5}{37M_{max}}\right\rvert$ $k = 1$ (lateral bending and warping are free) $= 0.5$ (lateral bending and warping are prevented) M_{max}—maximum bending moment M_1, M_5 are the bending moment at brace locations M_2, M_3, M_4—moments at the quarter-, mid-, and three-quarter points of the unbraced segment
Wong and Driver (2010) (AISC *Specification* Commentary Equation C-F1-2b)	$C_b = \frac{4M_{max}}{\sqrt{M_{max}^2 + 4M_A^2 + 7M_B^2 + 4M_C^2}} \leq 2.50$	M_{max}—absolute value of the maximum moment in the unbraced segment M_A, M_B, M_C—absolute values of the moments at the quarter-, mid-, and three-quarter points of the unbraced segment
* The moment modification factor in AASHTO *Specification* (2020) is expressed in the form of compression flange stresses at the brace locations.		

(AISC *Specification* Commentary Equation C-F1-5) (Yura, 1995; Yura and Helwig, 2010) for beams with reverse curvature continuously braced at their top flanges.

While the AISC *Specification* offers several equations that consider the moment gradient and the effect of bracing, there needs to be more discussion on the range of support and loading conditions for which the equations are applicable. Given the differences in the existing C_b formulae, particularly for end conditions with warping restraints, a general theoretical model based on the Rayleigh-Ritz formulation is presented in this paper to assess the accurate C_b for each loading and boundary condition. The frequent assertion in literature that using the C_b expressions derived for warping-free conditions is conservative when used in beams with warping restraint is also examined in this paper.

ELASTIC CRITICAL LTB CAPACITY USING THE RAYLEIGH-RITZ FORMULATION

This section provides the elastic critical LTB solutions obtained using the Rayleigh-Ritz method for beams with ideal boundary conditions subjected to different loading conditions. The standard case of a doubly symmetric I-beam with fork boundary conditions (flexurally and torsionally simply supported) subjected to uniform moments is shown in Figure 1. The traditional assumptions while deriving the LTB equation (Timoshenko, 1936) for I-shaped members are not stated here for brevity.

According to the principle of virtual work, the total potential, $\prod$, of a system may be determined by summing the elastic energy of the system, U, and the potential of the external forces, V. The total potential of the system is constant.

$$\prod = U + V \tag{3}$$

The total elastic strain energy of the beam is given by:

$$U = \frac{1}{2}\int_0^L \left[EI_y (u'')^2 + EI_w (\phi'')^2 + GJ(\phi')^2 \right] dz \tag{4}$$

And the potential of external force is equal to:

$$V = \frac{1}{2}\int_0^L M_x \left[2\phi (u'') \right] dz \tag{5}$$

The total potential of the system can hence be written as:

$$\prod = \frac{1}{2}\int_0^L \left[EI_y (u'')^2 + EI_w (\phi'')^2 + GJ(\phi')^2 \right] dz + \frac{1}{2}\int_0^L M_x \left[2\phi (u'') \right] dz \tag{6}$$

where

M_x = bending moment about the major principal axis

u'' = second derivative of the lateral deflection of the centroidal axis u

ϕ' = first derivative of the twist ϕ

ϕ'' = second derivative of ϕ

The potential energy is a function of the loading conditions (the bending moment M) and the unknown coefficients of the displacement functions (u and ϕ). These unknowns are calculated by applying the Rayleigh-Ritz technique to Equation 6 by minimizing the system's potential energy ($\delta\Pi = 0$).

Derivation of the elastic critical LTB capacity of a simply supported beam with fork boundary conditions subjected to uniform moment

The elastic critical LTB capacity for the doubly symmetric I-shaped member in Equation 1 is derived using the Rayleigh-Ritz approach described here.

The boundary conditions for a simply supported beam are given by

$$u = u'' = \phi = \phi'' = 0 \qquad \text{at } z = 0 \text{ and } z = L \tag{7}$$

The assumed displacement functions satisfying the above boundary conditions are:

$$u = A\sin\frac{\pi z}{L},\ \phi = B\sin\frac{\pi z}{L} \tag{8}$$

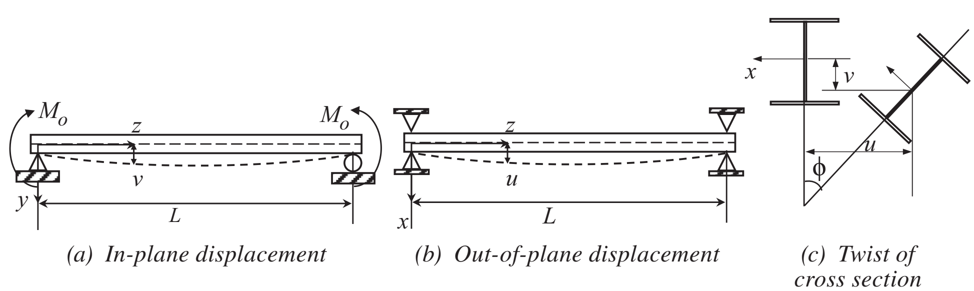


Fig. 1. Displacement of a simply supported doubly symmetric I-shaped member subjected to uniform moment.

The bending moment at any section along the length of the beam is $M_x = M$.

The total potential energy calculated using Equation 6 is given by

$$\Pi = U + V = \frac{\pi^4 EI_y A^2}{4L^3} + \frac{\pi^4 EI_w B^2}{4L^3} + \frac{\pi^2 GJB^2}{4L} - \frac{\pi^2 MAB}{2L} \tag{9}$$

Differentiating the total energy with respect to the unknowns A and B, the following equations are obtained.

$$\frac{\partial \Pi}{\partial A} = \frac{\pi^4 EI_y A}{2L^3} - \frac{\pi^2 MB}{2L} \tag{10}$$

$$\frac{\partial \Pi}{\partial B} = \frac{\pi^4 EI_w B}{2L^3} + \frac{\pi^2 GJB}{2L} - \frac{\pi^2 MA}{2L} \tag{11}$$

The total potential being constant, Equations 10 and 11 are equated to zero.

$$\begin{bmatrix} \frac{\pi^2 EI_y}{L^2} & -M \\ -M & \frac{\pi^2 EI_w}{L^2} + GJ \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = 0$$

The solution for the elastic critical lateral torsional moment, M_{cr}, can be obtained by evaluating the determinant of this matrix. Equation 12 is the same as the classical buckling solution derived by Timoshenko (1936), with an effective length factor, K, of 1.0.

$$M_{cr} = \sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)} \tag{12}$$

The different loading and boundary conditions studied in this paper are listed in Table 2. The in-plane boundary conditions and loading are illustrated through the images, and the warping restraint is described in the text. The transverse loads are applied at the centroidal axes, precluding instability from load-height effects.

Table 3 lists the elastic critical LTB capacities for the nine different loading and ideal boundary conditions listed in Table 2. These expressions are derived using the energy method. The cross-sectional twist is restrained at both beam ends in all cases in this paper. The assumed displacement functions listed in the last column of Table 2 satisfy the corresponding boundary conditions. The number of terms shown in the displacement functions is based on convergence studies for each case. The critical buckling load or moment is thus obtained by minimizing the potential energy given by Equation 6, using the same procedure outlined for a beam subjected to uniform moment (loading type 1).

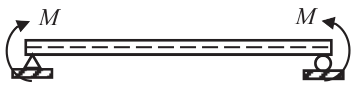
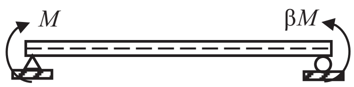
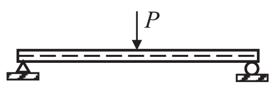
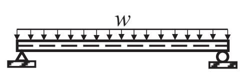
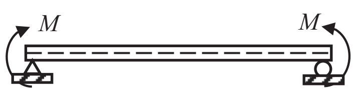
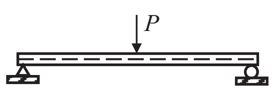
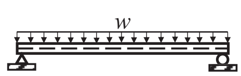
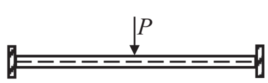
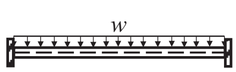
All critical moments are observed to be multiples of the basic critical moment, M_{ocr}, in Equation 1 with the corresponding K listed in Table 3 (1.0 for torsionally simply supported conditions and 0.5 for torsionally fixed conditions). The LTB modification factor, C_b, presented in Table 3, is the ratio of M_{cr} to M_{ocr}. The derivations for several cases listed in Table 3 are presented in Nayak et al. (2023). The derivations for loading types 6 and 9 are also presented in the appendix of this paper. The derivations for other cases are mathematically repetitive and can be obtained using the shape functions and boundary conditions listed in Table 2. They are not shown in this paper.

COMPARATIVE STUDIES WITH NUMERICAL RESULTS AND THE LITERATURE

The elastic critical LTB capacities calculated using the LTB modification factors derived in Table 3 are compared with the results from the elastic buckling analyses using FE simulations in SABRE2 (White et al., 2021). SABRE2 is a structural analysis and design software that employs beam elements with 7 degrees of freedom at each node, including the warping degree of freedom. The solutions from SABRE2 are also verified with several studies using elastic buckling analyses in ABAQUS (2022). Following a mesh convergence study, each beam unbraced segment is modeled with eight elements. The boundary conditions used are presented in Figure 2.

The results for two wide-flange sections (W16×40 and W30×90) of 6.0 m (19 ft 8 in) length are presented in this paper for the validation studies. Table 4 compares the critical elastic critical LTB capacities calculated using the LTB modification factors, C_b, established using the Rayleigh-Ritz method, $M_{cr,energy}$, and the elastic critical LTB capacities from the finite element analyses (FEA), $M_{cr,FEA}$, for the two sections for the 16 different cases that make use of results from the nine loading types chosen in Table 2. The results for a broader range of wide-flange sections are similar and do not add value to this paper. Cases 1–11 in Table 4 are flexurally and torsionally simply supported (twist restrained, warping free), while Cases 12–14 are flexurally simply supported in-plane, but torsionally fixed (twist and warping restrained). Cases 15 and 16 are modeled with both flexurally and torsionally fixed boundary conditions.

Figure 3 compares the elastic critical LTB capacities estimated using the empirical C_b equations given in the AISC *Specification* and in BS 5950-1 and those obtained from the energy method, with the elastic critical LTB capacities obtained from the FE simulations for the W16×40 and W30×90 sections, respectively. The values reported for the energy method employ $K = 0.5$ for conditions with warping fixity (Cases 12–16). In reporting the values for the AISC

Table 2. Assumed Displacement Functions for the Ideal Loading and Boundary Conditions Studied in This Paper				
		Boundary Condition		
Loading Type	Loading Condition	In-Plane Flexural	Warping	Displacement Functions
1		Simply supported	Free	$u = A\sin\frac{\pi z}{L}$ $\phi = B\sin\frac{\pi z}{L}$
2[a]		Simply supported	Free	$u = A\sin\frac{\pi z}{L} + B\sin\frac{2\pi z}{L}$ $\phi = C\sin\frac{\pi z}{L}$
3[b]		Simply supported	Free	$u = A\sin\frac{\pi z}{L} + B\sin\frac{2\pi z}{L}$ $\phi = C\sin\frac{\pi z}{L}$
4[b]		Simply supported	Free	$u = A\sin\frac{\pi z}{L} + B\sin\frac{2\pi z}{L}$ $\phi = C\sin\frac{\pi z}{L}$
5		Simply supported	Fixed	$u = A\left(1 - \cos\frac{2\pi z}{L}\right)$ $\phi = B\left(1 - \cos\frac{2\pi z}{L}\right)$
6[b]		Simply supported	Fixed	$u = A\left(1 - \cos\frac{2\pi z}{L}\right) + B\left(1 - \cos\frac{4\pi z}{L}\right)$ $\phi = C\left(1 - \cos\frac{2\pi z}{L}\right)$
7[b]		Simply supported	Fixed	$u = A\left(1 - \cos\frac{2\pi z}{L}\right) + B\left(1 - \cos\frac{4\pi z}{L}\right)$ $\phi = C\left(1 - \cos\frac{2\pi z}{L}\right)$
8[b]		Fixed	Fixed	$u = A\left(1 - \cos\frac{2\pi z}{L}\right) + B\left(1 - \cos\frac{4\pi z}{L}\right)$ $\phi = C\left(1 - \cos\frac{2\pi z}{L}\right)$
9[b]		Fixed	Fixed	$u = A\left(1 - \cos\frac{2\pi z}{L}\right) + B\left(1 - \cos\frac{4\pi z}{L}\right)$ $\phi = C\left(1 - \cos\frac{2\pi z}{L}\right)$

[a] β is positive for single curvature and negative for reverse curvature.
[b] The transverse loads are applied at the centroidal axes.

Table 3. Elastic Critical LTB Capacities and LTB Modification Factor Obtained for the Different Cases Considered in Table 2 Using the Energy Method			
Loading Type	***K***	**Elastic Critical LTB Capacity**	$C_b = M_{cr}/M_{ocr}$
1	1.0	$M_{cr} = \sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)}$	1.00
2	1.0	$M_{cr} = \frac{1}{\sqrt{\left[0.50(1+\beta)\right]^2 + \left[0.18(1-\beta)\right]^2}}\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)}$	$\frac{1}{\sqrt{\left[0.50(1+\beta)\right]^2 + \left[0.18(1-\beta)\right]^2}}$
3	1.0	$M_{cr} = 1.42\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)}$	1.42
4	1.0	$M_{cr} = 1.15\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)}$	1.15
5	0.5	$M_{cr} = \sqrt{\frac{\pi^2 EI_y}{(0.5L)^2}\left[\frac{\pi^2 EI_w}{(0.5L)^2} + GJ\right]}$	1.00
6	0.5	$M_{cr} = 1.07\sqrt{\frac{\pi^2 EI_y}{(0.5L)^2}\left[\frac{\pi^2 EI_w}{(0.5L)^2} + GJ\right]}$	1.07
7	0.5	$M_{cr} = 0.97\sqrt{\frac{\pi^2 EI_y}{(0.5L)^2}\left[\frac{\pi^2 EI_w}{(0.5L)^2} + GJ\right]}$	0.97
8	0.5	$M_{cr} = 1.08\sqrt{\frac{\pi^2 EI_y}{(0.5L)^2}\left[\frac{\pi^2 EI_w}{(0.5L)^2} + GJ\right]}$	1.08
9	0.5	$M_{cr} = 1.77\sqrt{\frac{\pi^2 EI_y}{(0.5L)^2}\left[\frac{\pi^2 EI_w}{(0.5L)^2} + GJ\right]}$	1.77

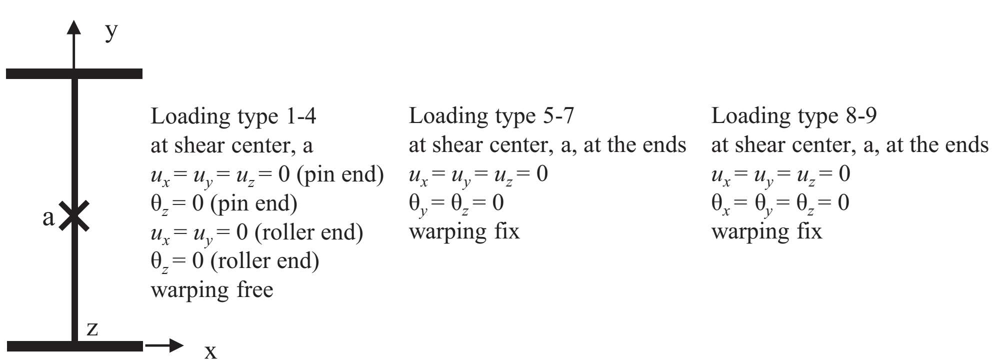


Fig. 2. Boundary conditions used in FE simulations.

Specification, a value of $K = 1.0$ is used in all cases, as per the current definition of L_b. Additionally, the AISC *Specification* C_b is also used with the correct effective length factor $K = 0.5$, for Cases 12–16, as proffered in the AISC *Specification* Commentary. Similarly, an effective length factor of 0.7 is employed in estimating the elastic critical LTB capacity while using BS 5950-1 for warping-fixed conditions.

Figure 4 compares the elastic critical LTB capacities estimated using the empirical equations in the literature and the energy-based solutions, with the elastic critical LTB capacities obtained from the FE simulations for the two sections. Nethercot and Rockey's (1972) equations were derived for warping-fixed support conditions. However, their direction on the appropriate effective length to use in their equations is ambiguous. Hence, an effective length factor of 1.0 is used here, even for cases with warping-fixed end conditions in their expressions for C_b. Using the full lateral unbraced length ($K = 1.0$) rather than an effective lateral length ($K = 0.5$) in their equations yields more realistic results, preventing grossly unconservative calculations. However, in other comparative studies, the critical moment is estimated using $K = 1.0$ for flexurally and torsionally simply supported boundary conditions and $K = 0.5$ for torsionally fixed end conditions irrespective of the in-plane flexural boundary conditions.

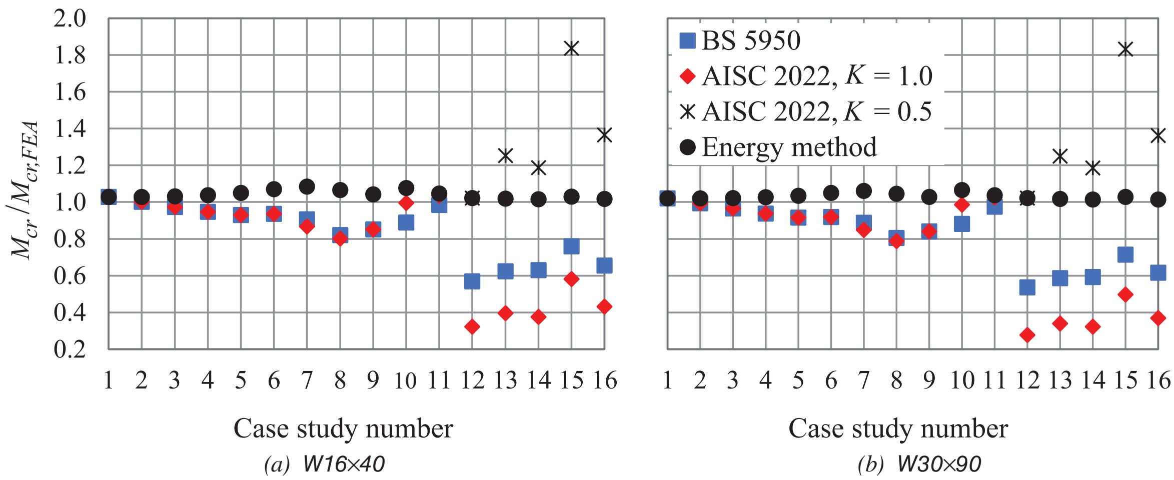


Fig. 3. Comparison of M_{cr} *obtained from FEA with the Rayleigh-Ritz method and equations given in the design standards for 6 m (19 ft 8 in) long I-shaped members.*

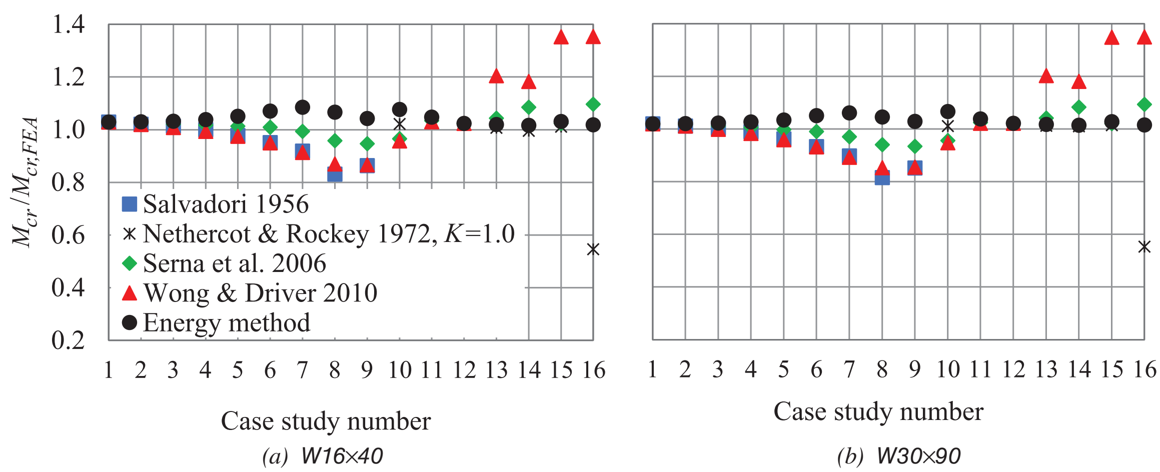


Fig. 4. Comparison of M_{cr} *obtained from FEA with the Rayleigh-Ritz method and empirical equations in literature for 6 m (19 ft 8 in) long I-shaped members*

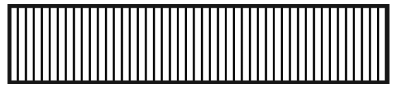
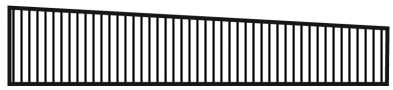
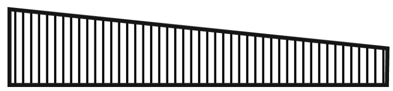
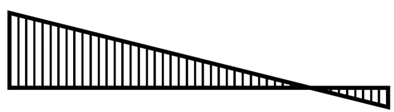
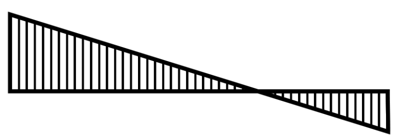
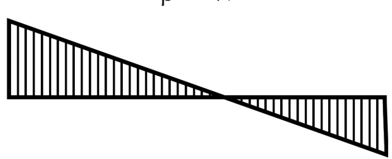
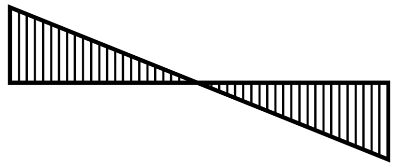
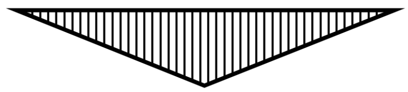
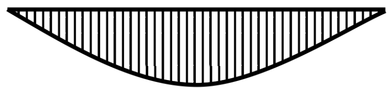
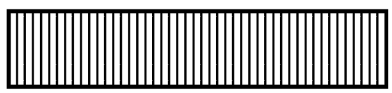
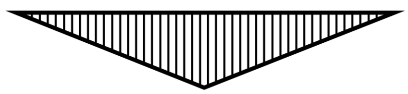
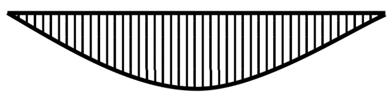
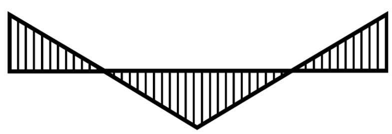
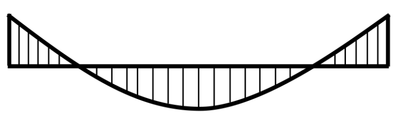
Case Study No.	Boundary Condition: Flexure	Boundary Condition: Warping	Bending Moment Diagram	W16×40 $M_{cr,FEA}/M_{cr,energy}$	W30×90 $M_{cr,FEA}/M_{cr,energy}$
1	Simply supported	Free	β = +1 	0.97	0.98
2	Simply supported	Free	β = +¾ 	0.97	0.98
3	Simply supported	Free	β = +½ 	0.97	0.98
4	Simply supported	Free	β = +¼ 	0.96	0.97
5	Simply supported	Free	β = 0 	0.95	0.97
6	Simply supported	Free	β = −¼ 	0.93	0.95
7	Simply supported	Free	β = −½ 	0.92	0.94
8	Simply supported	Free	β = −¾ 	0.94	0.96

Table 4. Comparison of the LTB Capacities Obtained from the Energy Method with FE Solutions

(Table 4 continues on the next page)

The following conclusions are drawn from Figures 3 and 4:

1. The calculated elastic critical LTB capacities for beams in the design code and specifications are typically conservative for torsionally simply supported end conditions, and the conservatism increases with an increase in the gradation of the moment. This is particularly true for the beams subjected to reverse curvature (Cases 6–9 in Figure 3). The elastic critical LTB capacity of a simply supported beam, calculated using the British standard (BS 5950-1, 2000) and the AISC *Specification* (2022) equations are nearly equal, and the maximum conservative strength estimations are 0.81 and 0.80 times $M_{cr,FEA}$, respectively, for Case 8 with an end moment ratio (β) of −0.75.

Table 4. Comparison of the LTB Capacities Obtained from the Energy Method with FE Solutions (continued)					
Case Study No.	Boundary Condition		Bending Moment Diagram	W16×40	W30×90
	Flexure	Warping		$M_{cr,FEA}/M_{cr,energy}$	$M_{cr,FEA}/M_{cr,energy}$
9	Simply supported	Free	$\beta = -1$ 	0.96	0.97
10	Simply supported	Free		0.93	0.94
11	Simply supported	Free		0.95	0.96
12	Simply supported	Fixed		0.98	0.98
13	Simply supported	Fixed		0.98	0.98
14	Simply supported	Fixed		0.99	0.99
15	Fixed	Fixed		0.97	0.97
16	Fixed	Fixed		0.98	0.98

2. The equations in the design code and specifications are overly conservative for beams with warping fixed at both beam ends (by 40–70%), when the comparisons use a K of 0.7 in the British standard and 1.0 in the AISC *Specification* equations. The British standard predicts elastic critical LTB capacities as low as 0.6 times those of the FE solutions for flexurally and torsionally fixed beams subjected to a uniformly distributed load (Case 16 in Figure 3). Similarly, the AISC *Specification* elastic critical LTB capacity is significantly conservative for beams with warping-fixed conditions (such as Case 15 in Figure 3), with a strength underestimation of up to 0.4−0.5 times the true solutions.

   On the other hand, if the comparison is made by including the true K of 0.5 for warping fixed conditions in the AISC *Specification*, such as in conditions with rigid beam-column joints, the strengths may be unconservatively estimated by up to 1.8 times the true strengths.

3. Figures 4(a) and (b) show that Salvadori's equation, although derived for linear moment gradients, conservatively estimates the elastic critical LTB capacity with a ratio of $M_{cr}/M_{cr,FEA}$ equal to 0.82 for Case 8 (with β equal to −0.75). Wong and Driver's equation underestimates the strength, with the smallest value of $M_{cr}/M_{cr,FEA}$ equal to 0.85 for simply supported beams with reverse curvature bending (Cases 8 and 9). Conversely, Wong and Driver's equation overestimates the strength in beams with warping-fixed end conditions,

with the largest $M_{cr}/M_{cr,FEA}$ ratio being 1.35 (in Cases 15 and 16), using K = 0.5. On the other hand, using K = 1.0 in Wong and Driver's equation for warping-fixed conditions will result in 50–70% conservative estimates of the true capacities. The excessively unconservative or conservative estimates of C_b shows the limitation of Wong and Driver's equation in warping-fixed end conditions.

4. The Nethercot and Rockey solutions estimate values of C_b greater than 3 for beams with warping fixity at their ends. Using K = 0.5 will make the equations unconservative, with the maximum value of the $M_{cr}/M_{cr,FEA}$ ratio equaling 3.19. Hence, these C_b factors are used along with an M_{cr} for an effective length factor of 1.0 rather than 0.5 to make the comparisons presented here more realistic. Figures 4(a) and (b) show that the solutions by Nethercot and Rockey (1972), derived for beams subjected to concentrated loads at their mid-spans and uniformly distributed loads, provide a reasonable estimate of M_{cr} for Cases 10 and 11 and Cases 13 and 14. All four of these cases are simply supported in-plane. Cases 10 and 11 are torsionally simply supported, while Cases 13 and 14 are torsionally fixed.

   The elastic critical LTB capacities for warping-fixed beams subjected to concentrated loads at their mid-spans are the same despite the difference in their in-plane flexural boundary conditions (i.e., in Cases 13 and 15). Consequently, ascribing the same LTB modification factor to Case 15 as Case 13 appears acceptable. However, the moment capacity of a flexurally and torsionally fixed beam subjected to a uniformly distributed load (Case 16) significantly differs from that of a flexurally simply supported and warping-fixed beam with the same loading scenario (Case 14). Therefore, using the same C_b for Cases 14 and 16 results in significantly conservative estimates of flexural strengths, with a ratio of $M_{cr}/M_{cr,FEA}$ of 0.55, suggesting that Nethercot and Rockey's equations for LTB modification factors for warping-fixed cases are better suited for in-plane simply supported boundary conditions, and are significantly conservative for Case 16, where the I-beam is flexurally fixed in-plane (resulting in reverse curvature, and torsional bracing at the location of the maximum moment within the unbraced span).

5. The equations by Serna et al. (2006) predict strengths that are typically smaller than the FEA results (up to 0.93 times $M_{cr,FEA}$ for simply supported beams with reverse curvature bending with β = −1, Case 9) and are unconservative (by approximately 10% of the FE solution) for a fully fixed beam subjected to a uniformly distributed load (Case 16). These equations, however, appear to provide the best estimates of C_b for the wide range of ideal loading and boundary conditions, even while exploiting the enhanced strengths from smaller effective lengths, and will be examined subsequently in this paper for laterally continuous beams.

6. Table 4 indicates that the LTB capacities estimated using the Rayleigh-Ritz method (denoted as the energy method) compare well with the FE results, with a maximum overestimation of 9% of $M_{cr,FEA}$ for a simply supported beam with moment gradient factor β equal to −0.5 (Case 7). These studies show that the energy formulations are most beneficial when applied to simply supported beams subjected to reverse curvature and for beams with warping fixed at their ends.

Impact of C_b on the limiting plateau length and the inelastic flexural strength

Having shown that the AISC *Specification* elastic critical LTB capacity is either overly conservative or unconservative for warping-fixed conditions depending on the assumed K, the import of applying the AISC *Specification* equations using K = 1 and 0.5, and C_b as per Equation F1-1 on the inelastic flexural strength and the plateau length is now examined. Figures 5(a) and (b) plot the normalized design strengths (M_n/M_p) for the flexurally and torsionally fixed beams with a concentrated load at the mid-span (Case 15, Table 4), and a uniformly distributed load (Case 16, Table 4) for the W16×40 section. The cross-section L_p is 1.7 m (5 ft 7 in), and L_r = 7.0 m (23 ft). These figures show the AISC *Specification* design strengths modified with the C_b as per Equation F1-1 and K = 1.0 and 0.5, and the design strengths estimated using the C_b factor derived using the energy method with a K = 0.5. On the other hand, the equations for C_b in the AASHTO *Specification* (Equations 6.10.8.2.3-7 and A6.3.3.7) recommend using an LTB modification factor of unity for cases where the compression flange stresses or the bending moments at the braced locations are zero. The AASHTO *Specification* commentary discusses that this would be conservative only in rare cases (in bridge girders) where the span is simply supported with no intermediate cross-frames. Figure 5 shows that using the AISC *Specification* C_b and K = 1.0 for warping fixed conditions is already on the safe side. Hence, the comparative study only includes the AISC *Specification* with the understanding that the use of the AASHTO *Specification* (with K = 1.0 and C_b = 1.0) will lead to still more conservative estimates of the beam capacity.

These plots further show that using a C_b factor as per the AISC *Specification* equation with an effective length factor of 1.0 results in a smaller plateau length and conservative estimates of design strengths when compared with the beam strengths estimated using the C_b factors derived from the energy method. It is important to note that the results presented in the paper are not specific to the two cross-sections shown, and any doubly symmetric hot-rolled section will produce similar results.

Using an inaccurate C_b along with $K = 0.5$ modifies the design strength curves such that the plateau lengths are 3.0 and 1.25 times those of the design curves obtained using the energy-based C_b factors for Cases 15 and 16, respectively. Furthermore, using an elastic K also inappropriately amplifies the inelastic LTB capacity.

Although the current AISC *Specification* equations sometimes lead to excessively conservative estimates of the flexural strengths when K is taken as 1.0 for warping fixed conditions, the degree of conservatism is such that such a simplification may not be suitable for design purposes. Conversely, using an effective length factor of $K = 0.5$ and the C_b factor from the current AISC *Specification* will result in highly unconservative estimates of flexural strengths. Therefore, any recommendation to use an effective length factor with the current C_b equations must be treated with caution.

APPLICABILITY OF THE LTB MODIFICATION FACTORS TO CONTINUOUS BEAMS

LTB modification factors are hitherto derived for ideal boundary conditions, which are simply supported and fixed flexurally and torsionally. However, even a single-span simply supported girder may have multiple lateral braces. The critical lateral span in such conditions receives partial restraint from its adjacent segments, and hence, the boundary conditions for the critical lateral span lie between the ideal simply supported and fixed conditions. In investigating the appropriate C_b, one must also use an appropriate effective lateral length factor, K. The K calculations in laterally continuous beams are more complex than those discussed with ideal boundary conditions.

Nethercot and Trahair (1976) were the first to suggest a method to estimate the effective length factor for beams, analogous to the buckling of braced columns, by accounting for the restraints from the adjacent spans. John and Subramanian found that the Nethercot and Trahair (N&T) method sometimes predicts significantly conservative or unconservative effective length factors. John and Subramanian contended that (1) the critical lateral span may receive restraint from segments further away from the immediate adjoining segments (the extended restraint effect); (2) the restraint received from the adjoining segment also depends on whether the brace is at the location of the maximum or the minimum moment within the critical span [resolved by the load and boundary condition effect (LBC)]; and (3) there are situations where the N&T method may incorrectly identify the critical lateral span and, thereby, the critical buckling load (resolved by an iteration of the N&T method). Three practical cases with laterally continuous spans are shown in Figure 6, for which the use of the appropriate C_b is discussed in this paper. The LBC interaction method proposed by John and Subramanian (2019) (referred to as J&S in this paper) is used in Cases I and II, while their iteration method is used in Case III.

In calculating M_{cr}, the different C_b formulations are combined with different K estimates. The C_b values from the proposed solutions and Serna et al. (2006) (previously shown to be the best estimates of C_b for a wide range of ideal boundary conditions) are used along with the elastic effective length factors derived using the classical N&T method and the modified method from J&S.

Figure 6 shows the loading and bending moment diagrams for the three cases discussed here. The beam is simply supported in-plane, and the lateral spans are marked as I, II, and III. The example discussed here considers a W30×90 beam section with a critical span length, L, of 6 m

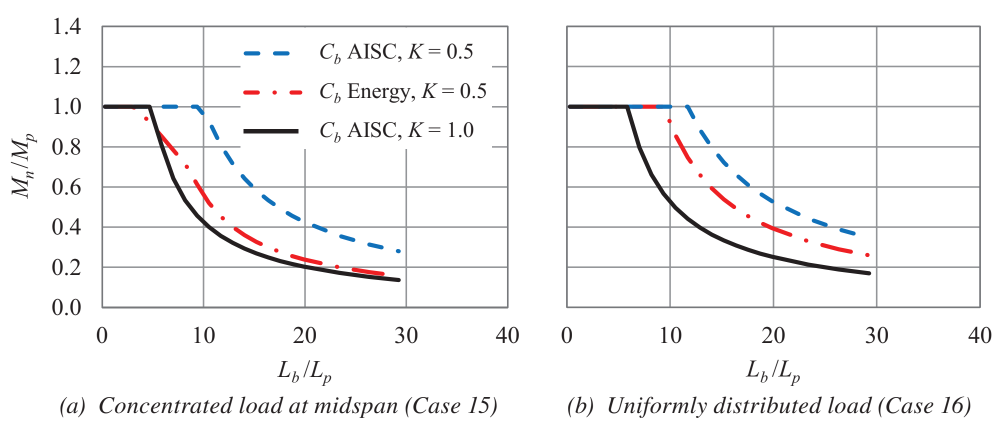


Fig. 5. Comparison of the design flexural strengths predicted using the C_b *factors in AISC* Specification *Equation F1-1 and the energy method for warping-fixed conditions for W16×40 section.*

(19 ft 8 in). The critical lateral span for this beam geometry in Case I is span I, and in Case II is span II. The critical lateral spans are marked by the hatched bending moment diagrams. The N&T method suggests that span II is critical in Case III, while span III is identified as the critical span using the iteration method (J&S) and FE test simulations. The difference in the critical spans identified by the two methods in Case III may be attributed to the restraint span II would receive at one end from span I, thereby enhancing its buckling strength. The increased elastic critical LTB capacity of span II will also result in greater restraint to the critical segment, span III.

Figure 7 shows the design flexural capacities for the example case by combining each of the two C_b formulations (from Serna et al., 2006, and the C_b evaluated using the energy method) with three different elastic effective length factors ($K = 1$, K from the N&T method, and K from the J&S method). The strengths are normalized by the true strengths from the FE test simulations. These calculations use the appropriate K in the C_b equations by Serna et al. instead of the binary values of 0.5 and 1.0 provided by the authors.

The following conclusions are drawn from Figure 7:

1. The flexural strengths calculated using C_b from the energy method and Serna et al., are conservative by 17–32% if K is taken as 1.0. The more conservative predictions are for Cases II and III, where the critical spans are braced by their adjoining segments at the locations of their maximum bending moments.
2. In Case I, where the adjacent span braces the smaller end moment location of the critical span, the N&T method, when used with the C_b from the energy method and Serna et al., estimates the elastic critical LTB capacities reasonably well. However, in Case II, with the larger end moment at the brace location, the estimated M_{cr} using the N&T method is conservative, with a ratio of $M_{cr}/M_{cr,FEA}$ equal to 0.87 for C_b from the energy method and 0.90 for C_b from Serna et al. In Case III, the estimated strengths exceed the FE solutions by 17% and 26% due to the incorrect identification of the critical span, as noted by John and Subramanian (2019).
3. The modified effective lengths proposed by John and Subramanian, along with the C_b from the energy method, predict the elastic critical LTB capacities with an error of less than 6% of the FE solutions across all cases.

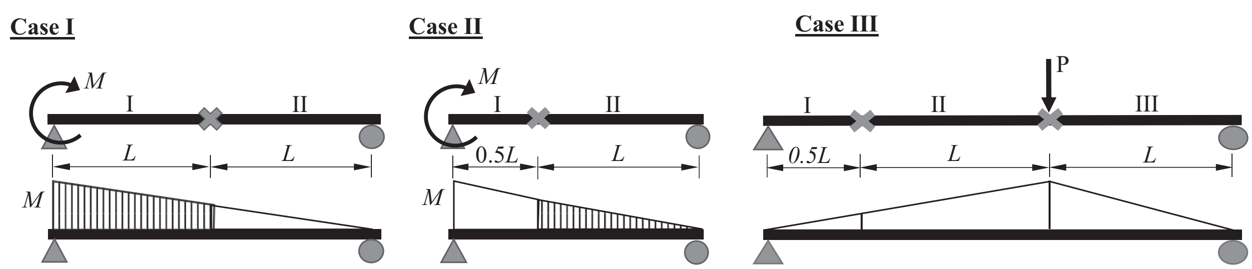


Fig. 6. Laterally continuous beams, where the critical lateral spans are indicated with a hatched bending moment diagram.

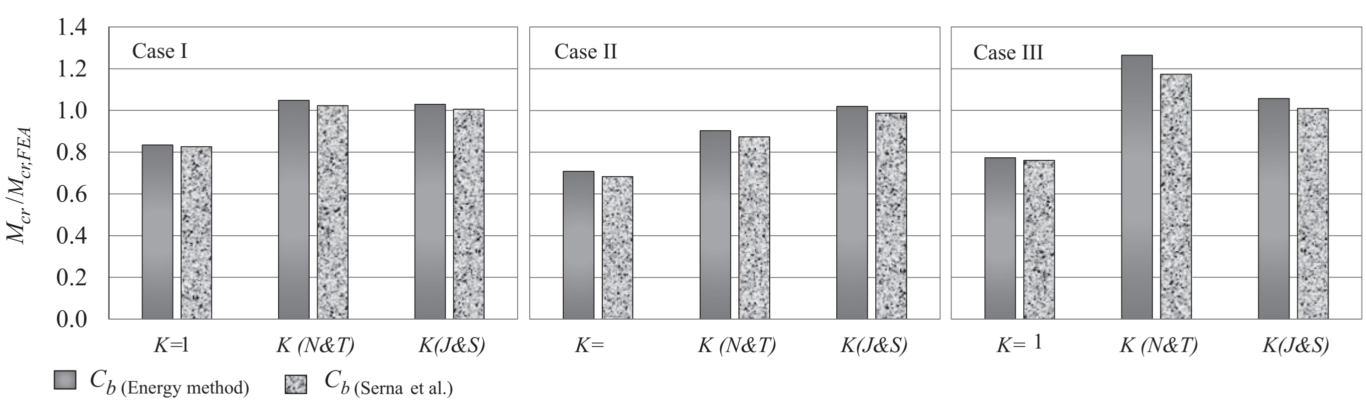


Fig. 7. Comparison of flexural strengths of the laterally continuous beams in Figure 6 using various C_b factors for W30×90, with L = 6 m (19 ft 8 in.)

4. In lieu of a rigorous formulation from the Rayleigh-Ritz method, the authors suggest using the C_b from Serna et al. (2006) in conjunction with the K from the J&S method, which best predicts the strengths of laterally continuous beams, with an error of less than 2%.

DISCUSSIONS ON PRACTICAL LOADING AND BOUNDARY CONDITIONS

The paper presents the behavior of I-shaped members subjected to transverse loads through the shear center. Many practical loading scenarios on I-section members in steel construction involve top flange loading, whereas gantry girders are typically loaded at their bottom flanges. A transverse load on an unbraced top flange reduces the elastic critical moment, while the load on an unbraced bottom flange increases the buckling moment. In such cases where the top flange is not braced, the load height effect must be considered in addition to the LTB modification factor, as also discussed in the AISC *Specification* Commentary. However, the presence of a concrete slab or deck precludes the tipping effect due to top flange loading. Members with ideal boundary conditions, such as simply supported and fixed flexurally and torsionally, are studied in this paper. The detailing of the beam-to-beam or beam-to-column connections dictates the warping fixity at the beam ends. For example, a simple shear connection is often torsionally simply supported at its ends. However, when a steel beam is encased into a concrete column, the column offers complete fixity against twisting and warping. Similarly, a moment connection offers significant restraint to warping and partially restrains in-plane and out-of-plane bending. Likewise, the secondary beams in a building often provide full or partial restraints to the primary beams at discrete locations, leading to the conditions of laterally continuous spans discussed in this paper. It is clearly important to be cognizant of the various in-plane and out-of-plane restraints that steel detailing provides and apply design equations accordingly.

CONCLUSIONS

This paper reviews the expressions for the LTB modification factor C_b for elastic lateral torsional buckling (LTB) of doubly symmetric I-shaped members in design codes and literature for different loading conditions, and different flexural and torsional boundary conditions. The paper includes combinations of ideal simply supported and fixed boundary conditions for flexure and torsion, as well as intermediate restraint conditions modeled using laterally continuous beams. The empirical equations for C_b are compared with analytical solutions derived by the authors using the Rayleigh-Ritz approach and finite element (FE) solutions. The key findings are summarized here:

1. The current equations in the AISC *Specification* (2022) and the British standard for C_b work well for flexurally and torsionally simply supported beams bending in single curvature, and are mildly conservative for such beams bending in reverse curvature. They are conservative by less than 10% when the beams are subjected to single curvature, and by less than 20% when they are subjected to reverse curvature bending.

2. AISC *Specification* Equation F1-1 is conservative by up to 268% for beams that are torsionally fixed and flexurally either simply supported or fixed, and loaded by a mid-span concentrated load and a uniformly distributed load (i.e., Cases 13–16). The equation is conservative for a beam with warping fixed at its ends and subjected to uniform moment at the ends (Case 12) by up to 72%. This is true if the full unbraced length (K = 1.0) is used. Further, the equations are excessively unconservative by up to 84% if an effective length factor of 0.5 is used as suggested in AISC *Specification* Section F1 Commentary. This would not be an ideal design solution for beams with laterally unbraced beams with moment connections at their ends, such as in rigid beam-column joints.

3. An inaccurate C_b results in falsely exaggerated plateau lengths and inelastic LTB capacities, especially when used together with an effective length factor, as discussed in the commentary of the AISC and AASHTO specifications.

4. The efficiency of the C_b factor is further examined using different K factors for laterally continuous beams. The energy method best predicts the moment modification factor, C_b, for ideal boundary conditions, although the C_b formulation from Serna et al. (2006) produces comparable results. However, when used in laterally continuous beams, one also must rely on empirical formulations to calculate the elastic effective length factor, K. For such beams, where the calculations combine an empirical K with a C_b, the authors find that the C_b from the energy method and Serna et al. when used along with the K from John and Subramanian (2019) produce similar results. The comparable predictions may be attributed to the approximate nature of the calculation methods for K. Recognizing the impracticality of the rigorous calculations involved in using the Rayleigh-Ritz approach for design, the authors recommend using the equations by Serna et al. for C_b along with effective length factors calculated using the methods by John and Subramanian.

Although the paper only presents the energy-based solutions for a few select cases, similar displacement shape functions may be used for other loading conditions to obtain the corresponding expressions for the LTB capacities. This offers a more rigorous approach to formulating the LTB modification factor than empirical fits to numerical data, especially for beams with warping fixity and reverse curvature. The procedure is outlined for a few typical loading and boundary conditions in the Appendix.

Despite the excessively conservative predictions when using the full unbraced lengths, the authors counsel against using an effective length factor for beams with warping restraints with the current C_b equations. The current AISC design equations work well for flexurally and torsionally simply supported laterally unbraced beams subjected to single curvature bending.

SYMBOLS

C_b Lateral torsional buckling modification factor for nonuniform bending moment diagrams

E Young's modulus of elasticity of steel, ksi

F_L Nominal compression flange stress above which the inelastic buckling limit states apply, ksi

F_{yc} Yield stress of compression flange, ksi

G Shear modulus of elasticity, ksi

I_w Warping constant, in.^{4}

I_y Moment of inertia of the cross section about the minor principal axis, in.^{4}

J St.-Venant torsional constant, in.^{4}

L, L_b Lateral unbraced length of the I-beam, in.

K Elastic effective length factor for lateral torsional buckling

$K_{inelastic}$ Inelastic effective length factor for lateral torsional buckling

L_p Limiting laterally unbraced length for the limit state of yielding, in.

L_r Limiting laterally unbraced length for the limit state of inelastic lateral-torsional buckling, in.

M Bending moment, kip-in.

M_{cr} Elastic critical lateral torsional buckling moment, kip-in.

$M_{cr,energy}$ Elastic critical lateral torsional buckling moment using the energy method, kip-in.

$M_{cr,FEA}$ Elastic critical lateral torsional buckling moment obtained from finite element simulations, kip-in.

M_n Nominal flexural strength as per the design specifications, kip-in.

M_{ocr} Elastic critical lateral torsional moment of beam subjected to uniform moment, kip-in.

M_p Plastic moment, kip-in.

P Magnitude of the transverse concentrated load, kips

P_{cr} Elastic critical buckling load, kips

U Elastic strain energy of the system

V Potential of the external forces

u Lateral/out-of-plane deflection, in.

w Magnitude of the uniformly distributed load, kip/in.

w_{cr} Elastic critical buckling load, kip/in.

Π Total potential of the system

ϕ Twist of the cross section

β Ratio of end moments, negative for reverse curvature bending

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REFERENCES

AASHTO (2020), *LRFD Bridge Design Specifications*, 7th Ed., Washington, D.C.

ABAQUS (2022), *Dassault Systémes*, Waltham, Mass.

AISC (2022), *Specification for Structural Steel Buildings,* ANSI/AISC 360-22*,* American Institute of Steel Construction, Chicago, Ill.

BS (2000), *Structural Use of Steelwork in Building. Part 1: Code of Practice for Design-Rolled and Welded Sections*, BS 5950-1, British Standards Institute, London, UK.

Galambos, T.V. (1998), *Guide to Stability Design Criteria for Metal Structures*, 5th Ed., John Wiley & Sons, New York, N.Y.

Galambos, T.V. and Surovek, A.E. (2008), *Structural Stability of Steel: Concepts and Applications for Structural Engineers,* Wiley & Sons, New York, N.Y.

John, J.B. and Subramanian, L.P. (2019), "A Modified Approach Towards Estimating the Lateral Torsional Buckling Effective Length," *Proceedings of the Annual Stability Conference*, SSRC, St. Louis, Mo., pp. 623–642.

Kim, Y.D. (2010), "Behavior and Design of Metal Building Frames Using General Prismatic and Web-Tapered Steel I-Section Members," Doctoral Dissertation, Georgia Institute of Technology, Atlanta, Ga.

Kirby, P.A. and Nethercot, D.A. (1979), *Design for Structural Stability*, John Wiley & Sons, New York, N.Y.

Nayak, N., Anilkumar, P.M., and Subramanian, L. (2023), "Rayleigh-Ritz Formulation for Moment Modification Factors on Lateral-Torsional Buckling of I-Beams," *Proceedings of the Annual Stability Conference*, SSRC, Charlotte, N.C., pp. 1–20.

Nethercot, D.A. and Rockey, K.C. (1972), "A Unified Approach to the Elastic Lateral Buckling of Beams," *Engineering Journal,* AISC, Vol. 9, No. 3, pp. 96–107.

Nethercot, D.A. and Trahair, N.S. (1976), "Lateral Buckling Approximations for Elastic Beams," *Structural Engineer*, Vol. 54, No. 6, pp. 197–204.

Salvadori, M.G. (1956), "Lateral Buckling of I-Beams," *Transactions of the American Society of Civil Engineers*, ASCE, Vol. 120, No. 1, pp. 1,165–1,177.

Serna, M.A., López, A., Puente, I., and Yong, D.J. (2006), "Equivalent Uniform Moment Factors for Lateral-Torsional Buckling of Steel Members," *Journal of Constructional Steel Research*, Vol. 62, No. 6, pp. 566–580.

Subramanian, L., Jeong, W.Y., Yellepeddi, R., and White, D.W. (2018), "Assessment of I-Section Member LTB Resistances Considering Experimental Test Data and Practical Inelastic Buckling Design Calculations," *Engineering Journal*, AISC, Vol. 55, No. 1, pp. 15–44.

Subramanian, L. and White, D.W. (2017a), "Reassessment of the Lateral Torsional Buckling Resistance of Rolled I-Section Members: Moment Gradient Tests," *Journal of Structural Engineering*, Vol. 143, No. 4, pp. 04016203.

Subramanian, L. and White, D.W. (2017b), "Reassessment of the Lateral Torsional Buckling Resistance of I-Section Members: Uniform-Moment Studies," *Journal of Structural Engineering*, Vol. 143, No. 3, pp. 04016194.

Timoshenko, S.P. (1936), *Theory of Elastic Stability*, 1st Ed., McGraw-Hill, New York, N.Y.

White, D.W. (2008), "Unified Flexural Resistance Equations for Stability Design of Steel I-Section Members: Overview," *Journal of Structural Engineering*, Vol. 134, No. 9, pp. 1,405–1,424.

White, D.W. and Jung, S. (2008), "Unified Flexural Resistance Equations for Stability Design of Steel I-Section Members: Uniform Bending Tests," *Journal of Structural Engineering*, Vol. 134, No. 9, pp. 1,450–1,470.

White, D.W., Tougay, O., Slein, R., and Jeong, W.Y. (2021), SABRE2, http://www.white.ce.gatech.edu/sabre.

Wong, E. and Driver, R.G. (2010), "Critical Evaluation of Equivalent Moment Factor Procedures for Laterally Unsupported Beams," *Engineering Journal*, AISC, Vol. 47, No. 1, pp. 1–20.

Yoo, C.H. and Lee, S.C. (2011), *Stability of Structures: Principles and Applications*, Elsevier, Oxford.

Yura, J.A. (1995), "Bracing for Stability—State-of-the-Art," *Proceedings of the ASCE Structures Congress XIII*, Boston, Mass., pp. 88–103.

Yura, J.A. and Helwig, T.A. (2010), "Buckling of Beams with Inflection Points," *SSRC Annual Stability Conference,* Orlando, Fla., pp. 761–780.

Ziemian, R.D. (2010), *Guide to Stability Design Criteria for Metal Structures*, 6th Ed., John Wiley & Sons, Hoboken, N.J.

APPENDIX: DERIVATION OF THE C_b FACTOR USING THE RAYLEIGH-RITZ METHOD

Loading type 6: simply supported beam with fork boundary conditions subjected to a concentrated load at the mid-span centroidal axis

The boundary conditions for a simply supported beam are given by

$$u = u'' = \phi = \phi'' = 0 \qquad \text{at } z = 0 \text{ and } z = L \tag{13}$$

The assumed displacement functions satisfying the above boundary conditions are

$$u = A\sin\frac{\pi z}{L} + B\sin\frac{2\pi z}{L},\ \phi = C\sin\frac{\pi z}{L} \tag{14}$$

The bending moment at any section along the length of the beam for a concentrated load P,

$$M_x = \begin{cases} \dfrac{Pz}{2} & 0 \le z \le \dfrac{L}{2} \\ \dfrac{P(L-z)}{2} & \dfrac{L}{2} \le z \le L \end{cases} \tag{15}$$

The total potential energy calculated using Equation 6 is given by

$$\Pi = \frac{\pi^4 EI_y(A^2+16B^2)}{4L^3} + \frac{\pi^4 EI_w C^2}{4L^3} + \frac{\pi^2 GJC^2}{4L} - \frac{(4+\pi^2)PAC}{16} \tag{16}$$

Differentiating the total energy with respect to the unknowns A, B, and C, the following equations are obtained.

$$\frac{\partial \Pi}{\partial A} = \frac{\pi^4 EI_y A}{2L^3} - \frac{(4+\pi^2)PC}{16} \tag{17}$$

$$\frac{\partial \Pi}{\partial B} = \frac{8\pi^4 EI_y B}{L^3} \tag{18}$$

$$\frac{\partial \Pi}{\partial C} = \frac{\pi^4 EI_w C}{2L^3} + \frac{\pi^2 GJC}{2L} - \frac{(4+\pi^2)PA}{16} \tag{19}$$

The total potential being constant, Equations 17–19 are equated to zero.

$$\begin{bmatrix} \frac{\pi^4 EI_y}{2L^2} & 0 & -\frac{4+\pi^2}{16}P \\ 0 & \frac{8\pi^4 EI_y}{L^3} & 0 \\ -\frac{4+\pi^2}{16}P & 0 & \frac{\pi^4 EI_w + \pi^2 GJL^2}{2L^3} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0$$

The solution for the elastic critical buckling load, P_{cr}, can be obtained by evaluating the determinant of the above matrix.

$$P_{cr} = \frac{8\pi^2}{L(\pi^2+4)} \sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)} \tag{20}$$

Now from the elastic critical lateral torsional buckling moment can be estimated using the expression $M_{cr} = \frac{P_{cr}L}{4}$

$$M_{cr} = 1.42\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)} \tag{21}$$

The LTB modification factor, C_b, is estimated as the ratio of M_{cr} given in Equation 21 and M_{ocr} (Equation 1) resulting in a value of 1.42.

Loading type 9: flexurally and torsionally fixed beam subjected to uniformly distributed load along the centroidal axis

The boundary conditions for a flexurally and torsionally fixed beam are given by

$$u = u' = \phi = \phi' = 0 \qquad \text{at } z = 0 \text{ and } z = L \tag{22}$$

The assumed displacement functions satisfying the above boundary conditions are

$$u = A\left(1-\cos\frac{2\pi z}{L}\right) + B\left(1-\cos\frac{4\pi z}{L}\right),\ \phi = C\left(1-\cos\frac{2\pi z}{L}\right) \tag{23}$$

The bending moment at any section along the length of the beam for a uniformly distributed load w per unit length,

$$M_x = \frac{w}{12}(6Lz - L^2 - 6z^2) \tag{24}$$

The total potential energy calculated using Equation 6 is given by

$$\Pi = \frac{4\pi^4 EI_y (A^2 + 16B^2)}{L^3} + \frac{4\pi^4 EI_w C^2}{L^3} + \frac{\pi^2 GJC^2}{L} - \frac{7LACw}{8} + \frac{11LBCw}{9} \tag{25}$$

Differentiating the total energy with respect to the unknowns A, B and C, the following equations are obtained.

$$\frac{\partial \Pi}{\partial A} = \frac{8\pi^4 EI_y A}{L^3} - \frac{7LCw}{8} \tag{26}$$

$$\frac{\partial \Pi}{\partial B} = \frac{128\pi^4 EI_y B}{L^3} + \frac{11LCw}{9} \tag{27}$$

$$\frac{\partial \Pi}{\partial C} = \frac{8\pi^4 EI_w C}{L^3} + \frac{2\pi^2 GJC}{L} - \frac{7LAw}{8} + \frac{11LBw}{9} \tag{28}$$

The total potential being constant, Equations 26–28 are equated to zero.

$$\begin{bmatrix} \frac{8\pi^4 EI_y}{L^3} & 0 & -\frac{7Lw}{8} \\ 0 & \frac{128\pi^4 EI_y}{L^3} & \frac{11Lw}{9} \\ -\frac{7Lw}{8} & \frac{11Lw}{9} & \frac{8\pi^4 EI_w + 2\pi^2 GJL^2}{L^3} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0$$

The solution for the elastic critical buckling load, w_{cr}, can be obtained by evaluating the determinant of the above matrix.

$$w_{cr} = \frac{144\pi^2}{\sqrt{4453}L^2} \sqrt{\frac{\pi^2 EI_y}{(0.5L)^2}\left[\frac{\pi^2 EI_w}{(0.5L)^2} + GJ\right]} \tag{29}$$

Now, the elastic critical lateral torsional buckling capacity can be estimated using the expression $M_{cr} = \frac{w_{cr}L^2}{12}$

$$M_{cr} = 1.77\sqrt{\frac{\pi^2 EI_y}{(0.5L)^2}\left[\frac{\pi^2 EI_w}{(0.5L)^2} + GJ\right]} \tag{30}$$

The LTB modification factor, C_b, is estimated by taking the ratio of M_{cr} given in Equation 28 and M_{ocr} for warping-fixed condition (Equation 1) with effective unbraced length $0.5L$, resulting in a value of 1.77.