Yield-Line Analysis and Design of Grids

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INCREASING ATTENTION has been given in recent years to design methods which recognize and account for the reserve strength which often exists in a structure beyond the occurrence of first fiber yielding.[1,2,3] These methods, variously named, use the structure's collapse behavior as the basis for proportioning. Service load checks must then be made on alternative failure modes, such as deflection.

Most applications of collapse design have been confined to structures composed of "line elements," e.g., beams and frames. Reinforced concrete designers have been given guarded approval to extend collapse analysis to slabs, if such analysis is accompanied by verifying load tests.[4] Thus the so-called "yield-line" approach affords a powerful slab preliminary design tool which, adequately supported by experimental verification, can be used to determine final proportioning of slab and reinforcing steel.[5] A yield-line type of analysis can also be employed to appraise the collapse strength of steel grid systems. This paper presents yield-line analysis and design procedures for orthogonal grid systems. Confirming laboratory tests are also presented and discussed.

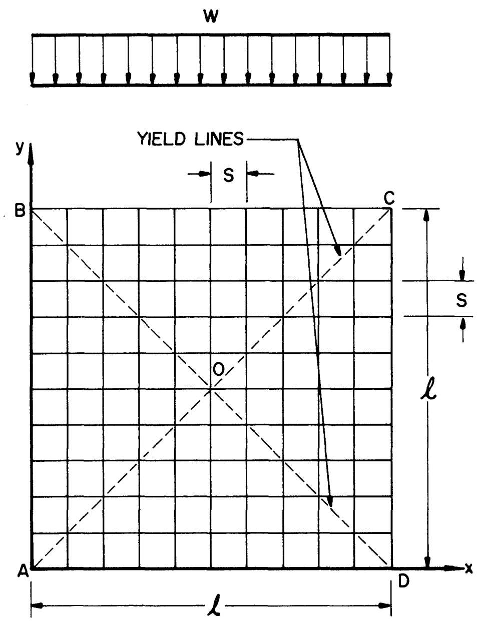

Fig. 1. Square grid and yield-line pattern for edge supports and uniform load

GRID YIELD-LINE ANALYSIS

Assume an orthogonal steel grid formed of single size flexural members, each having plastic moment capacity M_p. Initially, the members are assumed to form a square grid with equal spacings, s, in the x- and y-directions (Fig. 1). The grid is simply supported on all four sides. For increasing gravity-type load, the grid members will eventually develop extreme fiber yield stresses in the vicinity of the center of the grid. The grid is not near collapse. Moment redistribution occurs extensively in this highly redundant structure as more load is applied. Finally, plastic hinges form in individual beams as the applied moment reaches M_p, the resisting plastic moment value. Successively formed plastic hinges form a yield-line, analogous to the yield-line formed in a reinforced concrete slab. The fully developed yield-lines will be substantially straight lines coinciding with the diagonals **AC** and **BD** of the grid. Application of additional load beyond that required for full yield-line development will result in a flexural collapse of the grid accompanied by large deflections. The load associated with the impending grid collapse is analogous to the load associated with the collapse of a beam or frame structure.

The grid collapse load, w_p, can be determined by considering the equilibrium, at collapse, of a segment of the grid, say **ABOA**, which is shown as a free-body diagram in Fig. 2. Conditions of equilibrium and symmetry require that the total slab reaction along support **AB** balance the applied vertical load, $wl^2/4$, which acts on a quarter of the grid. No net shear force can exist on **AO** and **BO**. Equilibrium of moments produced about axis **AB** involves only the applied load and the resisting plastic moments, M_{py}. For $\Sigma M_{AB} = 0$:

$$\frac{w_p l^2}{4}\left(\frac{l}{6}\right) - M_{py}(n) = 0 \qquad (1)$$

where n is the number of beams spanning in the y-direction cut by the **AO** and **BO** yield lines. The number of

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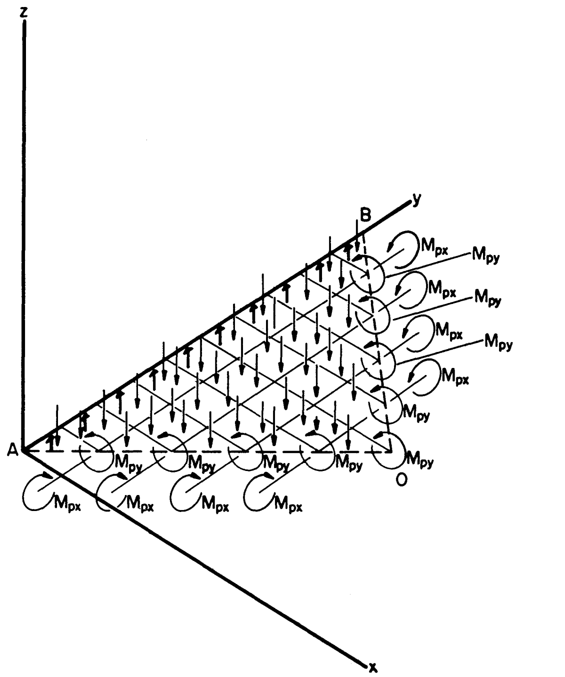

Fig. 2. Free-body diagram of segment ABOA of square grid

beams, n, will be $[(l/s) - 1]$, from which Eq. (1) can be written

$$w_p = \frac{24M_p\left(\frac{l}{s} - 1\right)}{l^3} \tag{2}$$

Moment equilibrium considerations for the other three similar segments will confirm Eqs. (1) and (2). Grids continuous over their supports would develop negative bending moments which could be accounted for in the foregoing equilibrium equations. If the full plastic moment, M_p, were developed along the supports, Eq. (2) would become

$$w_p' = \frac{48M_p\left(\frac{l}{s} - 1\right)}{l^3} \tag{2a}$$

For non-square grids and/or unequal x and y beam spacing and/or unequal x and y beam size, the yield-lines associated with collapse will not, in general, coincide with the grid diagonals. Instead, the pattern will be as shown in Fig. 3, where X is to be determined. To determine the collapse pattern and load of this more general grid, the equilibrium of two segments, labeled as ① and ② in Fig. 3, must be examined. For segment ① (**ABFA**), equilibrium of moments about **AB** requires:

$$w_1(l_y)\left(\frac{l_x - X}{2}\right)\frac{1}{2}\left(\frac{l_x - X}{2}\right)\frac{1}{3} = n_y M_{py} \tag{3}$$

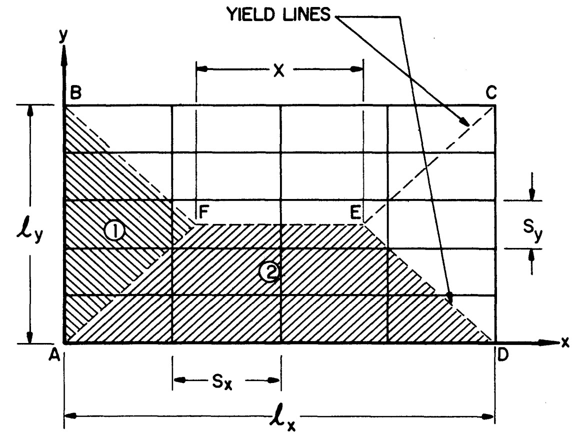

Fig. 3. General rectangular grid showing yield-line pattern

or

$$w_1 = \frac{24\left(\frac{l_y}{s_y} - 1\right)M_{py}}{(l_x - X)^2\, l_y} \tag{4}$$

For segment ② (**AFEDA**), moment equilibrium about **AD** leads to

$$w_2(X)\left(\frac{l_y}{2}\right)\left(\frac{l_y}{4}\right) + w_2(l_x - X)\left(\frac{l_y}{2}\right)\frac{1}{2}\left(\frac{l_y}{6}\right) = n_x M_{px} \tag{5}$$

or

$$w_2 = \frac{24\left(\frac{l_x}{s_x} - 1\right)}{l_y^2(l_x + 2X)}\, M_{px} \tag{6}$$

For a correct yield line pattern, i.e., the pattern actually associated with collapse, $w_1 = w_2 = w_p$. Trial values of X and successive solutions of Eqs. (4) and (6) will lead quickly to the correct collapse pattern and load.

The foregoing analysis does not account for the effects of "corner levers" and "equivalent nodal forces".[5] These concepts account, respectively, for branching of a yield line as it approaches a corner and the moment associated with shears other than zero along yield lines. These factors, while acknowledged to exist, are thought to be an unnecessary refinement in this presentation. The effect of inclusion of these factors seems to be small for usual support and loading conditions.

DESIGN

For given spans and loading, Eqs. (4) and (6) can be used as grid design equations. The parameters for which design is sought would usually be grid spacing and member plastic moment capacity. Member spacing and moment capacity can be combined into one design variable, the unit plastic moment capacity, m_p. In a manner like that for slab design this is defined as a moment per unit length of grid. Along a particular line the unit moment would be

$$m_p = \frac{nM_p}{l} \tag{7}$$

Applying Eq. (7) to Eqs. (4) and (6) (for $w_1 = w_2 = w_p$) leads to

$$m_{py} = \frac{w_p(l_x - X)^2}{24} \tag{8}$$

and

$$m_{px} = \frac{w_p l_y^2(l_x + 2X)}{24 l_x} \tag{9}$$

These two equations in three unknowns (m_{py}, m_{px}, and X) can be solved only by assuming a value for one variable or some one relationship among the variables. Typically, the relationship between m_{py} and m_{px} would be known or could be approximated. Due to the nature of the equations, successive solutions for m_{py} and m_{px} for trial values of X is often most expeditious.

Design Example—Assume it is required to design a steel grid to span a 50-ft square with a design live load of 80 lbs/ft^2. If structural bar joists are employed in a "thread through" grid arrangement, it might be expected that the shallow joists would be of the order of two-thirds the depth of the joists through which they are threaded. If it is assumed that the joists have equal chord sections, the moment resistances will be proportional to the depths. For a dead load of 20 lbs/ft^2 and a load factor of 2.0, the collapse load, w_p, is to be 200 lbs/ft^2 = 0.2 kips/ft^2. From Eqs. (8) and (9),

$$\frac{m_{px}}{m_{py}} = \frac{l_y^2(l_x + 2X)}{l_x(l_x - X)^2} \tag{10}$$

For $m_{px}/m_{py} \approx 3/2$ solve Eq. (10) for X by trial. Assume $X = 8$ ft.

$$\therefore \frac{m_{px}}{m_{py}} = \frac{(50)^2\,(50 + 16)}{50\,(50 - 8)^2} = \frac{25\,(11)}{21\,(7)}$$
$$= 1.87 > 1.5$$

One can deduce the direction X must go to get m_{px}/m_{py} closer to the desired value. The value of $X = 8$ ft corresponds to $m_{px}/m_{py} = 1.87$, but the desired moment ratio is 1.5. Examination of Fig. 3 and the moment equilibrium about **AB** and **AD** of segments 1 and 2, respectively, leads us to conclude that segment 1 must be made larger and segment 2 smaller to decrease the moment ratio m_{xy}/m_{py}. Accordingly, try $X = 2$ ft.

$$\frac{m_{px}}{m_{py}} = \frac{(50)^2\,(50 + 4)}{50\,(50 - 2)^2} = \frac{50\,(54)}{48\,(48)} = \frac{75}{64} = 1.17 < 1.5$$

The value of X has now been reduced too much. A linear interpolation between $X = 8$ ft and $X = 2$ ft leads to

$$\frac{1.87 - 1.17}{8 - 2} = \frac{1.5 - 1.17}{X - 2} \text{ or } X \cong 5 \text{ ft.}$$

Using $X = 5$ ft in Eq. (10) leads to

$$\frac{m_{px}}{m_{py}} = \frac{(50)^2\,(50 + 10)}{50\,(50 - 5)^2} = \frac{50\,(60)}{(45)\,(45)} = 1.48 \approx 1.5$$

To determine the required values of m_{px} and m_{py}, Eqs. (8) and (9) can be used.

$$m_{py} = \frac{0.2(50 - 5)^2}{24} = \frac{0.2(45)\,(45)}{24} = 16.88 \text{ kips}$$

Since $m_{px}/m_{py} \approx 1.5$,

$$m_{px} \approx 1.5\,(16.88) = 25.32 \text{ kips}$$

To confirm, use Eq. (9) to get $m_{px} = 25$ kips.

As an alternative to the foregoing trial and error analysis, one can expedite analysis and design through use of the graphs of Fig. 4. The curves represent solutions of Eq. (10) which can be expressed as

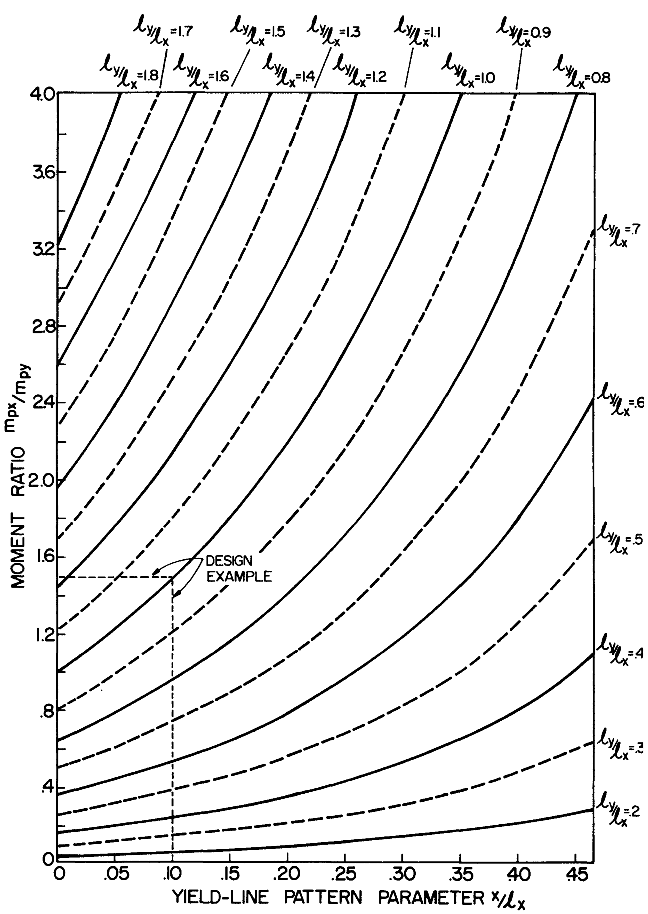

Fig. 4. Analysis and design curves for general rectangular grid

$$\frac{m_{px}}{m_{py}} = \frac{\left(\frac{l_y}{l_x}\right)^2\left(1 + \frac{2X}{l_x}\right)}{1 - \frac{2X}{l_x} + \left(\frac{X}{l_x}\right)^2} \tag{11}$$

For a given ratio of rectangle sides, l_y/l_x, the unit moment capacity ratio, m_{px}/m_{py}, is a function only of the yield line "pattern" expressed as the ratio X/l_x. Various ratios of l_y/l_x are represented parametrically in Fig. 3(b). The use of Fig. 3b for design will be illustrated by the previous example. For an assumed moment ratio, m_{px}/m_{py}, of 1.5 and a square grid (i.e., $l_y/l_x = 1.0$) the correct collapse yield line pattern is seen from Fig. 4 to be defined as $X/l_x = 0.1$, from which $X = 0.1\ (50) = 5$ ft. Individual required values of unit moments, m_{px} and m_{py}, are then determined directly from Eqs. (8) and (9):

$$m_{py} = 0.2\,\frac{(50 - 5.0)^2}{24} = 16.88 \text{ kips}$$

and

$$m_{px} = 0.2\,\frac{(50)^2\,[50 + 2(5.0)]}{24(50)} = 25.0 \text{ kips}$$

For a structural bar joist, the plastic moment capacity M_p, is approximately given by

$$M_p = A_c d \sigma_y \tag{12}$$

where

A_c = area of one (smallest) chord member
d = effective depth (measured to centroid of chords)
σ_y = yield stress

Assuming joists on 4-ft centers (both ways), one obtains, from $M_p = sm_p$, M_{px}(req'd) = 4 (25) = 100 kip-ft; M_{py}(req'd) = 67 kip-ft.

A 24H8 joist ($\sigma_y = 50$) has a bottom chord area of 1.103 in.^{2} and an effective depth of 23.106 in., so its plastic moment capacity is 1.10 (23.10)(50) = 1270 kip-in. = 106 kip-ft > 100. A 16H8 joist moment capacity is 15.1 (1.1)(50) = 830 kip-in. = 69 kip-ft > 67. The complete preliminary design then calls for 16H8 joists threaded through 24H8 joists, both sets of joists spaced on 4-ft centers. This design, of course, would have to be checked for service load deflections, buckling, and shear. Design associated with grids having other support and/or loading conditions than those presumed in this example can be similarly accomplished after establishment of the correct yield lines. Procedures which have been developed for slabs are applicable.[5]

LOAD TESTS

Load tests were conducted on two grid systems. Both were composed of bar-joist-like members. In the first, a model spanning 12.5 ft in each direction was constructed

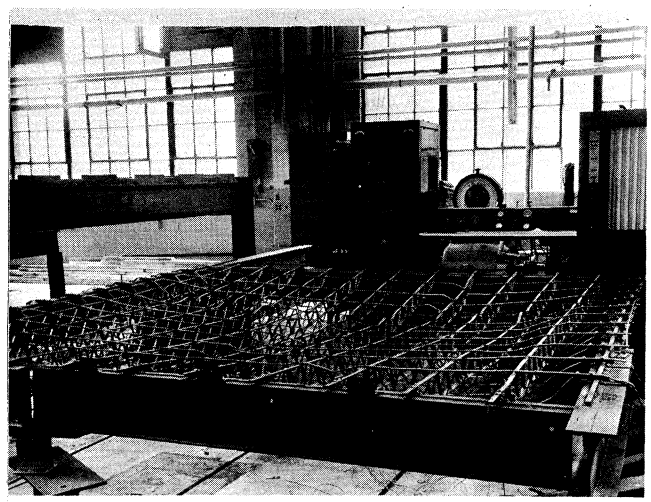

Fig. 5. Thread-through model grid system after completion of test

from mild steel flat strip stock (Fig. 5). Top chords were of 3⁄16 × ¾-in. stock, while bottom chords were 1⁄16 × 1-in. Material in chords and web approximated ASTM A36 steel in behavior. Joists of 4-in. depth were threaded through 6-in. joists. Both sets of joists were spaced 12 in. apart. Chord-web connections were made with stove bolts.

The values of unit-width plastic moment capacity resulting from this arrangement were $m_{px} = 0.9$ kips and $m_{py} = 0.6$ kips. Moment capacity reduction due to the holes in the chord members is accounted for in these values. The predicted collapse pattern of this grid uniform load could be ascertained by using the curves from Fig. 4. For $l_y/l_x = 1.0$ and $(m_{px}/m_{py}) = 3/2$, $X/l_x = 0.1$. From Eq. (8),

$$w_p = \frac{0.60\ (24)}{(12.5 - 1.25)^2} = 0.115 \text{ kips/ft}^2 \text{ or } 115 \text{ lbs/ft}^2.$$

This structure was load-tested with a uniform load created by a water-filled plastic membrane. The load-deflection behavior is plotted in Fig. 6. The failure, occurring at a load just slightly less than the predicted collapse load, was one of yielding accompanied by some inelastic buckling of the compression chords (Fig. 5).

The second load test was conducted on a steel grid fabricated from real structural bar joists. Six modified 8H2 joists of 17.5 ft span, were orthogonally threaded through six 12H2 joists of 17.5-ft length. The plastic moment capacities of each joist, calculated from Eq. (12), are:

8H2: $M_p = 0.378(6)(50) = 113$ kip-in.
12H2: $M_p = 0.378(10.2)(50) = 195$ kip-in.

$$m_{px} = \frac{195(6)}{17.5(12)} = 5.6 \text{ kips};\ m_{py} = \frac{113(6)}{17.5(12)} = 3.2 \text{ kips}$$

$$\frac{m_{px}}{m_{py}} = \frac{5.6}{3.2} = 1.75$$

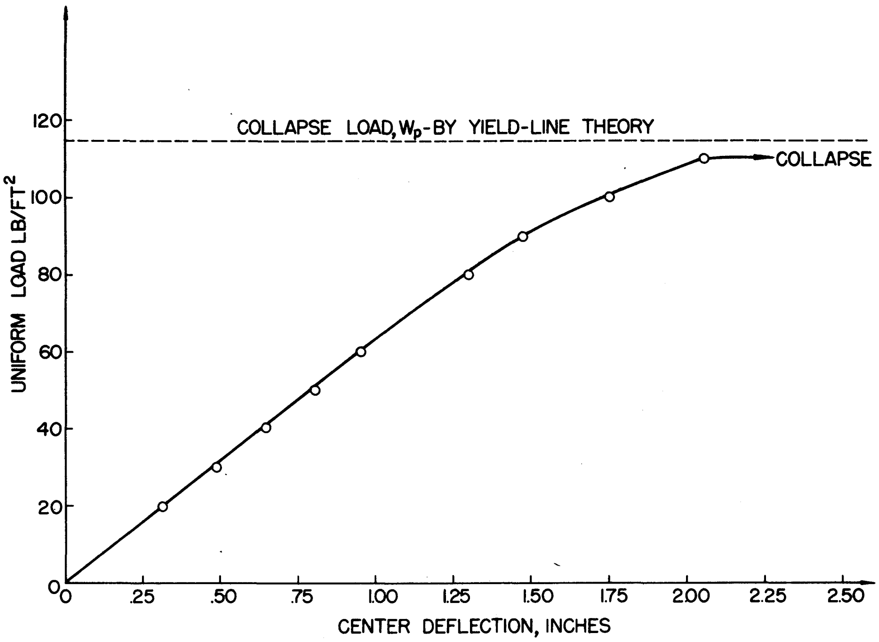

Fig. 6. Load-deflection curve for thread-through model grid

From Fig. 4:

$$\text{For } l_y/l_x = 1.0, X/l_x = 0.145$$

$$X = 0.145(17.5) = 2.5 \text{ ft}$$

From Eq. (8):

$$w_p = \frac{24(3.23)}{(17.5 - 2.5)^2} = 0.340 \text{ kips/ft} = 340 \text{ lbs/ft}^2$$

The load test of this system was carried out in a manner similar to that used for the model grid system. The load vs. grid center deflection is shown in Fig. 7. In this test, threatened failure of the loading system dictated termination of the test at a load of 260 lbs/ft^2. This load, while short of the theoretical collapse load of 340 lbs/ft^2, was of sufficient magnitude to demonstrate the synergistic effect of the two way grid. These same members spanning one direction only have a theoretical collapse load of 230 lbs/ft^2. It would appear from examination of Fig. 7 that a load of approximately 300 lbs/ft^2, or 90 percent of the predicted failure load, could have been sustained before collapse. The results of these two load tests indicate actual collapse loads of the order of 90–100 percent of the theoretical collapse load.

USE IN DESIGN

It would appear that grids have less reserve strength than slabs. Whereas slabs invariably exhibit greater strength than that predicted by yield-line theory,[5] the tests reported herein seem to indicate that for grids the theoretical yield-line collapse load provides a good, yet upper bound, measure of the true collapse load. The reasons for greater slab than grid strength are not completely obscure. The presence of "holes" in the grid structure prevents development of plate action including the strengthening effect of Poisson's ratio and twisting moments. The foregoing suggests that a grid proportioned on the basis of the yield-line approach will be somewhat less conservative than a slab, assuming identical load factors. Alternatively, equally conservative grid and slab designs can be realized by assigning different load factors. For preliminary design, load factors of 2.3 and 2.0 for grid and slab, respectively, are thought

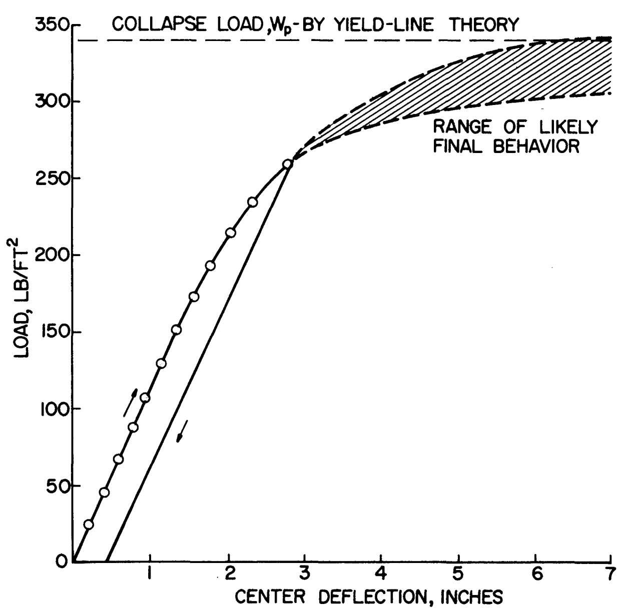

Fig. 7. Load-deflection curve for thread-through bar joist grid

by this writer to be reasonable. It is thought that these load factors would provide safety factors against a strength failure of at least 2.0 for either grids or slabs.

ACKNOWLEDGMENT

The author acknowledges the help of his students in pursuing this study. Those assisting in fabricating and testing the two grids reported in this paper, were Gene Davis, Anthony Benincasa, Charles Crouse, Edward Dufton, Lloyd Ackerman, William Daigneau, and Dean Palmer, all seniors in civil engineering at Case Western Reserve University at the time the tests were conducted. Acknowledgment is also given to Bethlehem Steel Corporation, and to Republic Structural Iron Works for supplying materials and fabrication for the bar joist test.

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