Analysis of Curved Steel Girder Bridges

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IN RECENT YEARS, with the introduction of highway interchanges in cities, a demand for curved bridges has developed. A typical city interchange is one proposed for Interstate Highway 95 in Providence, R. I. (see Fig. 1), for which most of the ramps are on structure. Steel girder bridges are a good solution to this type of problem.

Many such horizontally curved bridges have been built, but generally the steel girders have been straight. However, if one knew with reasonable certainty how to design curved steel girders, there would be definite advantages. Four of these are shown in Fig. 2:

(1) Fewer piers are required, because straight girders (which are really chords of the bridge curve) have to be relatively short.

(2) The curved girders can be made continuous, making the use of shallower sections possible.

(3) The slab overhang from the girder farthest from the center of curvature is substantially less than for the straight girders.

(4) The appearance of curved girders is much better than straight ones when the curvature is appreciable.

A research project, "Circular-Arc I-Beam Highway Bridges," sponsored by the Rhode Island Department of Public Works in cooperation with the United States Bureau of Public Roads, is being conducted at the University of Rhode Island. This project, started in 1963 and recently extended, includes among its objectives:

(a) The development of a theoretical analysis for a curved girder bridge
(b) The testing of several model bridges to provide comparisons with the theory
(c) The development of a simplified design procedure for structures of this type

It is the purpose of this paper to present an outline of the method of analysis. The testing phase of the project is discussed by Professor C. Bernard Clarke in his paper[1] elsewhere in this Journal.

The analysis is based on the general stiffness method of analysis of elastic structures.[2, 3] For the purposes of this paper, a stiffness matrix K is defined by the matrix equation $W = K\Delta$, where W and Δ are column matrices of forces and deflections respectively at s directions of interest, and K is a square matrix of order s.

The symbols adopted for use in this paper are defined where they first appear.

THE METHOD OF ANALYSIS

A curved girder bridge is treated as a planar grid structure loaded normal to its plane. Figure 3a shows such a structure, greatly simplified for academic reasons. A right-handed coordinate system is established; this is called the *structure system* and is denoted by the superscript o. Joints are established at supports and wherever members frame together; these are numbered consecutively from 1 to n (where n is the number of joints), in any convenient order. The members are numbered consecutively from 1 in any convenient order.

Each vector of interest (force or deflection) at any joint has three components, one linear and two angular. These components are always considered in the order: linear along Z-axis, angular about X-axis, and angular about Y-axis; a component being positive when its sense agrees with that of the corresponding reference axis. An angular vector is represented by a double-headed linear vector normal to its plane, according to the right-hand rule (i.e., in the direction of the extended thumb of the right hand when the fingers of the right hand curl in the direction of rotation). The directions of interest are established by numbering the component directions consecutively from 1 to $3n$, following the standard order at each joint taken in numerical order. Thus, as shown in Fig. 3b, the directions of interest at any joint j are $3j$-2, $3j$-1, and $3j$.

For any loading applied to the structure, three equations of equilibrium are written for each joint. These can be expressed as one matrix equation, which, assuming

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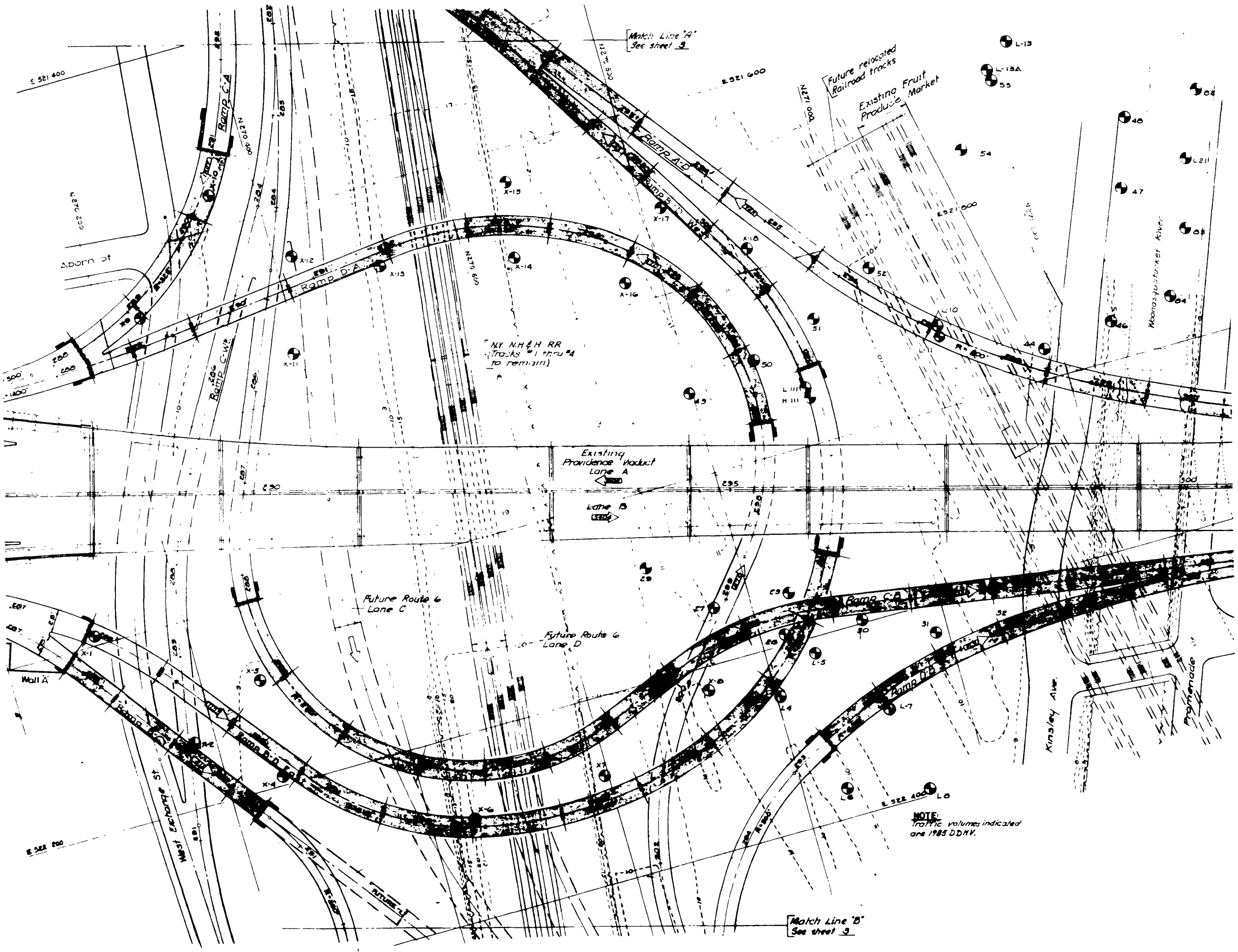


Fig. 1. Proposed Providence Interchange

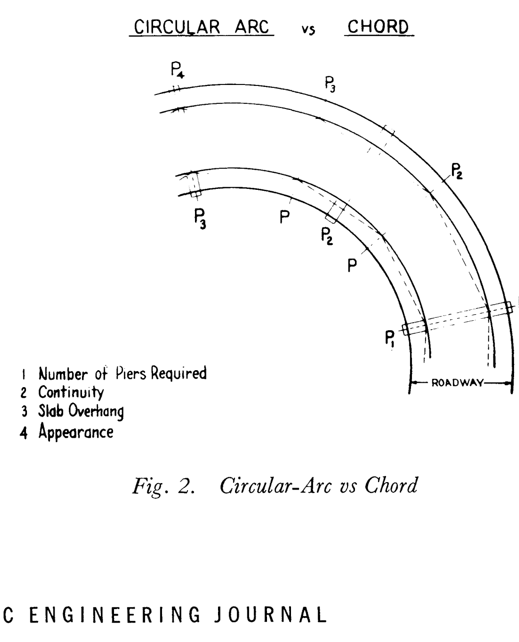


Fig. 2. Circular-Arc vs Chord

that loads are applied only at joints, takes the form:

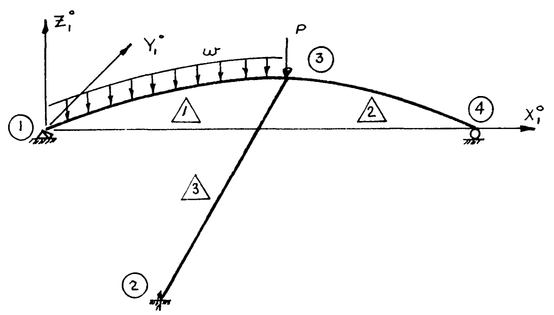
$$W^o + R^o - K^o\Delta^o = 0 \qquad (1)$$

where

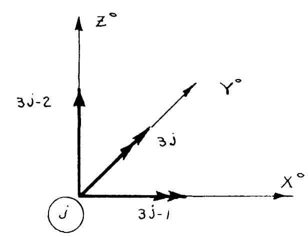
K^o = Structure stiffness matrix
W^o = Structure load matrix
R^o = Structure reaction matrix
Δ^o = Deflection matrix

W^o, R^o, and Δ^o are column matrices of order $(3n)$ x 1 with their components being in the order of the directions of interest, and K^o is a symmetrical matrix of order $3n$.

Two mutually exclusive lists of directions of interest are prepared, the *r-list* of restrained directions (i.e., directions in which free deflections are prevented by reaction components) and the *u-list* of unrestrained directions, such that any direction of interest appears in only one of the two lists. If there are r restraints, then u =

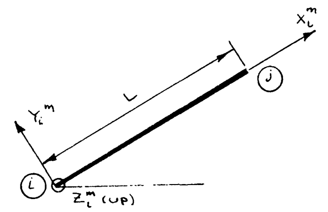


a. Joint and Member Numbering

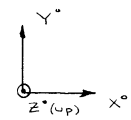


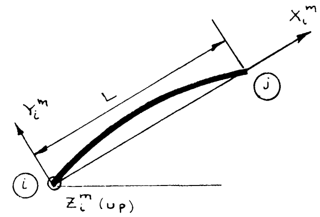
b. Direction Numbering

Fig. 3. Numbering Schemes



a. Straight member





b. Curved member

Fig. 4. Member Reference Systems

$3n - r$ and Equation (1) represents $3n$ equations in terms of r unknown reactions and u unknown deflections. To solve this equation, it is separated into an equivalent pair of matrix equations:

$$(uW)^o - (uKu)^o(u\Delta)^o = 0 \tag{2}$$

and

$$(rW)^o + (rR)^o - (rKu)^o(u\Delta)^o = 0 \tag{3}$$

where the pre-modifier u (or r) implies a matrix formed from the parent matrix by using only the rows matching the u-list (or r-list), and the post-modifier u implies a matrix formed by using only the columns matching the u-list.

Equation (2) is solved to obtain the unrestrained deflection matrix:

$$(u\Delta)^o = [(uKu)^o]^{-1} (uW)^o \tag{4}$$

which is then substituted in Equation (3) to obtain the restrained reaction matrix:

$$(rR)^o = (rKu)^o(u\Delta)^o - (rW)^o \tag{5}$$

From these results, any other desired information is obtained readily.

MEMBER STIFFNESS MATRIX

Each member has its own stiffness matrix relating internal forces and moments at its ends to the deflections at the ends. Thus, for a member framing between joint i and joint j, we have:

$$M_{ij}{}^m = K_{ij}{}^m \Delta_{ij}{}^m \tag{6}$$

or

$$\begin{bmatrix} M_i{}^m \\ M_j{}^m \end{bmatrix} = \begin{bmatrix} k_{ii}{}^m & k_{ij}{}^m \\ k_{ji}{}^m & k_{jj}{}^m \end{bmatrix} \begin{bmatrix} \Delta_i{}^m \\ \Delta_j{}^m \end{bmatrix} \tag{6a}$$

where:

$M_a{}^m$ = The 3 components of internal forces at end a (shear in Z^m-direction, torque about X^m-axis, moment about Y^m-axis)

$\Delta_a{}^m$ = The 3 components of deflection at a

$k_{ab}{}^m$ = 3 x 3 stiffness submatrix relating $M_a{}^m$ to $\Delta_b{}^m$; and the superscript m denotes the member reference system (related to the end joints) in which all components are expressed. See Fig. 4

The elements in the above submatrices depend on the properties of the material (modulus of elasticity, modulus of rigidity), the cross-sectional properties (area, moment of inertia, the torsional constant), and the geometry of the member (length, curvature, variation in cross-section). Formulas are available for prismatic straight members[4] and for prismatic circular-arc members;[5] values for other cases can be found by numerical procedures.

By suitable coordinate transformations,[6] it is possible to modify Equation (6) to relate vectors in other coordinate systems:

$$M_{ij}{}^m = K_{ij}{}^{mo}\Delta_{ij}{}^o \tag{6b}$$

and

$$M_{ij}^{o} = K_{ij}^{omo} \Delta_{ij}^{o} \tag{6c}$$

From the last equation, we can write:

$$K_{ij}^{omo} = \begin{bmatrix} k_{ii}^{omo} & k_{ij}^{omo} \\ k_{ji}^{omo} & k_{jj}^{omo} \end{bmatrix} \tag{7}$$

where the triple superscript identifies the reference systems involved in the transformations.

STRUCTURE STIFFNESS MATRIX

The structure stiffness matrix K^o of Equation (1) is assembled by summing the submatrices of the individual member stiffness matrices as defined in Equation (7). Thus,

$$K^o = \begin{bmatrix} k_{11}^o & k_{12}^o & . & . & . & k_{1n}^o \\ k_{21}^o & k_{22}^o & . & . & . & k_{2n}^o \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ k_{n1}^o & k_{n2}^o & & & & k_{nn}^o \end{bmatrix} \tag{8}$$

where

$k_{ii}^o = \Sigma k_{ii}^{omo}$ for all members framing into joint i

$k_{ij}^o = k_{ij}^{omo}$ for the member framing between joints i and j ($k_{ij}^o = 0$, if no member frames between the joints)

LOAD SYSTEMS

For loads applied to a member ij between its ends, the fixed-end forces (including moments and torques) required at the ends to prevent any joint displacement are determined[7] and expressed in the member system as FM_{ij}^m and in the structure system as FM_{ij}^o.

For any loading condition on the structure, two new load matrices are defined: FW^o, obtained by summing the fixed-end forces at every joint produced by loads applied to members between the joints; and JW^o, consisting of those loads applied directly to the joints. These are column matrices with components in the order of the directions of interest.

The loading condition is resolved into two load systems: a fixed-end system, consisting of the FW^o matrix and all the loads applied between joints; and a joint-loads-only system, consisting of the loads:

$$W^o = JW^o - FW^o \tag{9}$$

The solution for the first load system is obtained readily since each member is not influenced by the loads on any other member, and the solution for the second system is obtained by applying Equations (1) to (5). The final results are obtained by superposition of these two solutions.

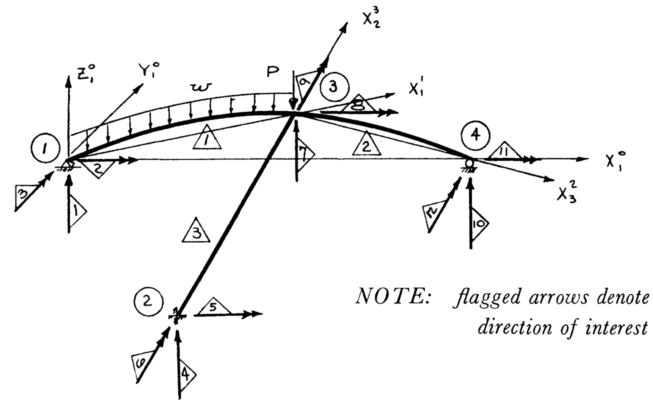


Figure 5

APPLICATION OF METHOD

The application of the method is illustrated by outlining the analysis for the academic structure of Fig. 3a.* This structure is shown again in Fig. 5 with the member reference systems and numbering of directions of interest added.

1. *Generation of member stiffness matrix:*
   Member 1, from joint 1 to joint 3:
   Compute K_{13}^1, K_{13}^{10} and K_{13}^{010}
   Member 2, from joint 3 to joint 4:
   Compute K_{34}^2, K_{34}^{20} and K_{34}^{020}
   Member 3, from joint 2 to joint 3:
   Compute K_{23}^3, K_{23}^{30} and K_{23}^{030}
   Assemble member stiffness matrix:

$$K^o = \begin{bmatrix} k_{11}^{010} & 0 & k_{13}^{010} & 0 \\ 0 & k_{22}^{030} & k_{23}^{030} & 0 \\ k_{31}^{010} & k_{32}^{030} & (k_{33}^{010} + k_{33}^{020} + k_{33}^{030}) & k_{34}^{020} \\ 0 & 0 & k_{43}^{020} & k_{44}^{020} \end{bmatrix}$$

2. *Modify stiffness matrix:*
   From the indicated support restraints (vertical restraint only at joints 1 and 4, and fully fixed at joint 2), form:
   r-list = 1, 4, 5, 6, 10 (5 restrained directions)
   u-list 2, 3, 7, 8, 9, 11, 12 (7 unrestrained directions)
   From K^o, form:
   $(uKu)^o$ a 7 x 7 matrix, using rows and columns matching u-list
   $(rKu)^o$ a 5 x 7 matrix, using rows matching r-list and columns matching u-list
   Evaluate $[(uKu)^o]^{-1}$

* *The detailed solution for this problem is given in* A Program to Analyze Curved Girder Bridges, *by Francis H. Lavelle and John S. Boick, University of Rhode Island Division of Engineering Research and Development, Engineering Bulletin No. 8, December 1965, p. 35–53.*

3. *For the loading shown:*

Compute FM_{13}^{1} and FM_{13}^{o} for member 1 (the only loaded member) and form FW^{o}

Form JW^{o} and $W^{o} = JW^{o} - FW^{o}$

From W^{o}, form:

$(uW)^{o}$, using elements matching u-list

$(rW)^{o}$, using elements matching r-list

From Equation (4): $(u\Delta)^{o} = [(uKu)^{o}]^{-1}(uW)^{o} =$ deflections in unrestrained directions

From Equation (5): $(rR)^{o} = (rKu)^{o}(u\Delta)^{o} =$ reactions in restrained directions

Expand $(u\Delta)$ to obtain:

$$\Delta^{o} = \begin{bmatrix} \Delta_1^{o} \\ \Delta_2^{o} \\ \Delta_3^{o} \\ \Delta_4^{o} \end{bmatrix}$$

Obtain end forces in each member using Equation (6b):

Member 1: $$\begin{bmatrix} M_1^{1} \\ M_3^{1} \end{bmatrix} = K_{13}^{10} \begin{bmatrix} \Delta_1^{o} \\ \Delta_3^{o} \end{bmatrix} + \begin{bmatrix} FM_1^{1} \\ FM_3^{1} \end{bmatrix}$$

Member 2: $$\begin{bmatrix} M_3^{2} \\ M_4^{2} \end{bmatrix} = K_{34}^{20} \begin{bmatrix} \Delta_3^{o} \\ \Delta_4^{o} \end{bmatrix}$$

Member 3: $$\begin{bmatrix} M_2^{3} \\ M_3^{3} \end{bmatrix} = K_{23}^{30} \begin{bmatrix} \Delta_2^{o} \\ \Delta_3^{o} \end{bmatrix}$$

The results for the curved members can be transformed into tangential reference systems.

COMPUTER PROGRAM

The above method has been programmed for a digital computer, using FORTRAN IV,[8, 9] to handle any configuration of circular-arc and straight members of constant sections connected rigidly at joints.

The input consists of:

(a) Coordinates of each joint
(b) For each member: its joint numbers, section properties, and if curved, the radius and central angle
(c) The r-list of restrained directions
(d) For each load condition: joint loads, fixed-end forces for each loaded member

The output consists of:

(a) Deflections and rotations at each joint
(b) All reaction components
(c) The shears, moments, and torques at each end of all members

Variations in this program have been written for the IBM 1410, IBM 7094, Burroughs B5500 and Univac 1107 computers. Curved girder bridges have been solved having up to 269 members and 150 joints (involving inversion of matrices up to order 426) with apparent success.

ACKNOWLEDGMENT

The computer program was developed at the University of Rhode Island as part of the project "Circular-Arc I-Beam Highway Bridges," sponsored by the Rhode Island Department of Public Works and the United States Bureau of Public Roads. The project is directed by Professors C. Bernard Clarke and Francis H. Lavelle, Co-Principal Investigators.

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4. *Ibid., page 274.*
5. *Francis H. Lavelle and John S. Boick* A Program to Analyze Curved Girder Bridges, *University of Rhode Island, Division of Engineering Research and Development, Engineering Bulletin No. 8, December, 1965, pages 7 to 9.*
6. *Ibid., pages 9 and 10.*
7. *Ibid., pages 11 to 13.*
8. *Ibid., pages 16 to 34.*