On the Lateral Support of Inelastic Columns

GEORGE PINCUS

THE FAILURE buckling load of centrally loaded columns with a high slenderness ratio is given by the Euler elastic buckling formula:

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{1}$$

If the column fails at an average stress which is higher than the proportional limit of the material (defined as the yield stress less the maximum residual stress), then the failure load becomes,[1, 2, 3]

$$P_{cr} = \frac{\pi^2 E_T I}{L^2} \tag{2}$$

where E_T is the tangent modulus of elasticity. In either case, the failure load is directly proportional to the moment of inertia about an axis perpendicular to the direction of buckling, I, and to the tangent modulus or elastic modulus of elasticity, whichever case applies.

IDEALIZED COLUMN MODEL

A formula similar to equation (2) above may be obtained from a simple column model which has been used by Shanley, Winter, and others. The ideal column consists of two infinitely rigid portions ($EI = \infty$) connected by a rotational spring of stiffness β representing the overall flexural stiffness of the real column (see Fig. 1). If the ends are pinned as shown and axial loads P act on both ends, then the ideal column deflects as shown. In Fig. 1, the moment at the center is:

$$M = \beta(2\theta) \tag{3}$$

Summing moments about either end of half the column,

$$Pd = M \tag{4}$$

but

$$\theta = \frac{d}{L/2} = \frac{2d}{L} \tag{5}$$

George Pincus is Assistant Professor of Structural Engineering, Department of Civil Engineering, University of Kentucky, Lexington, Kentucky.

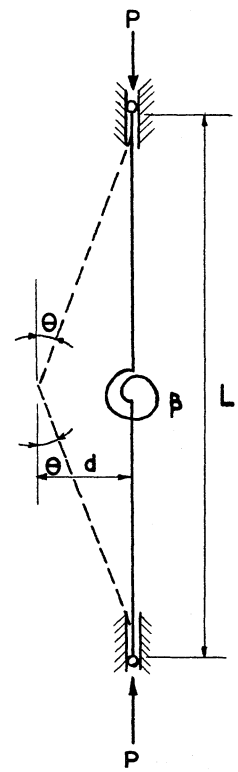

Fig. 1. Idealized centrally loaded column

for small deflections, so that substituting (4) and (5) into (3)

$$P = \frac{4\beta}{L} \tag{6}$$

Substituting equation (6) in equation (1), β of the equivalent system can be determined for the condition when the load P is the critical load:

$$\beta = \frac{\pi^2 EI}{4L} \tag{7}$$

If failure occurs within the elastic range of the material, the spring constant β is seen to be dependent on

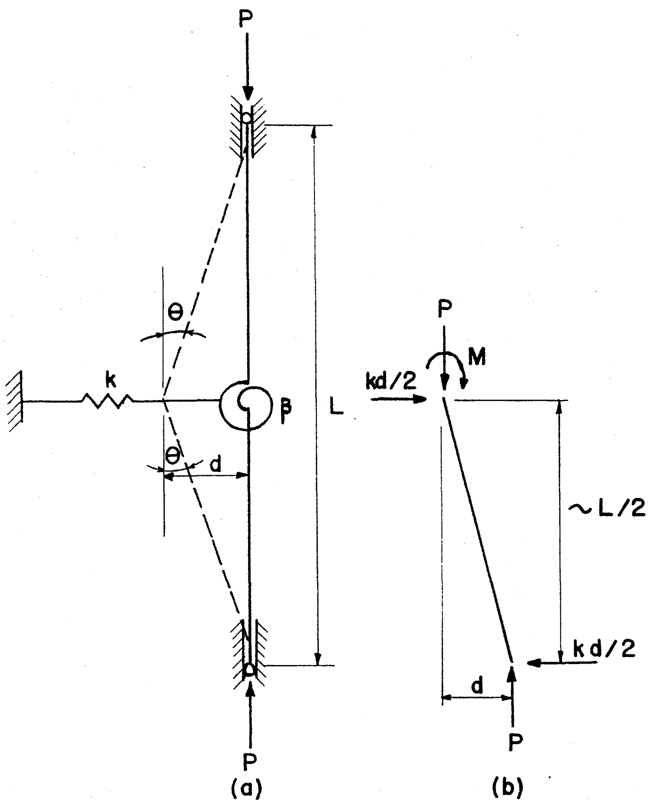

Fig. 2. Ideal column with lateral support

a fixed value of the modulus of elasticity E. If, on the other hand the material is inelastic at failure, E would be replaced by the tangent modulus E_T and β would then become implicitly dependent on the inelastic failure load.

IDEALIZED COLUMN MODEL WITH LATERAL SUPPORT

It may be shown, with the use of such an idealized column model, that if the column is laterally supported throughout its length, the required stiffness of the lateral support for the column to reach its yield load is much greater if the section fails at an inelastic load than if it fails while the material is fully elastic. In Reference 4, the failure load of an elastic centrally loaded column is given as

$$P_{\text{Fail}} = P_{\text{Euler}} + Q \tag{8}$$

where P_{Euler} is the failure load for the unsupported column (elastic) and Q is a term representing the shear rigidity of the lateral support.

For the inelastic column, the variation of the parameters which define the material properties of the column such as flexural rigidity, twisting rigidity and warping rigidity must be recognized. These vary as loading progresses and inelasticity is attained and also along the length of the column. In addition, the rigidity of the lateral support, Q, may be affected if the column material is inelastic. A straightforward solution such as expression (8) above cannot be obtained and numerical methods have to be used. Thus, the procedure of finding the failure load for an inelastic column is tedious and cumbersome. In view of this difficulty in obtaining an exact theoretical solution for the inelastic column, the idealized column model is used.

Referring to Fig. 2a, an additional linear spring of stiffness k represents the restraint of the lateral support against buckling. As the load P is applied to the ends of the column, one-half of the restraining linear spring force is applied to each segment of the column and in the free body of Fig. 2b

$$M = Pd - \frac{kdL}{4} \tag{9}$$

but the moment in the rotational spring is

$$M = 2\theta\beta \tag{10}$$

and again

$$\theta = \frac{d}{L/2} \tag{11}$$

for small deflections.

Using equations (10) and (11) in (9), the following equation is obtained:

$$P = \frac{4\beta}{L} + \frac{kL}{4} \tag{12}$$

It is interesting to compare equation (12) with equation (8). In both, the first term represents the elastic failure load of the unbraced column; the second term is the contribution of the support.

In equation (12), if β starts to decrease, i.e., E begins to drop (equation (7)), as loading progresses, then the failure load P will be smaller than if β were to remain constant. Thus, the failure load will be smaller for the inelastic condition than if the member were to remain elastic up to failure. If it is required that the failure load reach a fixed value, which is in the inelastic range, then, once β starts to decrease (start of inelasticity), failure can be prevented by increasing k. Therefore, if failure is to take place in the inelastic range, a larger amount of (or stiffer) lateral support is required.

Since some designers base the size or stiffness of lateral support on a certain percentage of the main member area or load, the procedure may lead to unsafe results in short inelastic columns. The rule of thumb of providing lateral support with some percentage of the strength of the column may be a perfectly safe (and probably overconservative) method to attain full lateral support if the column is fully elastic at failure. However, care must be taken not to apply the "rule of thumb" to columns failing within the inelastic range, lest unsafe results arise. Since many building columns are of such l/r ratios that the critical load would be reached in the inelastic range, the importance of this phenomenon to the design engineer is obvious.

SUMMARY

An idealized column model may be used to find the buckling load of a centrally loaded column. A linear spring may be added to such a model to represent the effect of lateral support on the column. It is shown that inelasticity at failure requires a greater amount of (or stiffer) lateral support to reach a given load than that based on elastic conditions at failure. This phenomenon must be recognized by designers who assume lateral support as fully effective when it meets an arbitrary criterion or rule of thumb.

ACKNOWLEDGMENT

Some of the concepts employed in this article were developed during the investigation *The Performance of Columns and Beams Continuously Braced with Diaphragms*, Gordon P. Fisher, Project Director, and George Winter, Project Consultant, sponsored at Cornell University by the American Institute of Steel Construction. The use of the idealized restrained column model to represent the laterally supported column was first suggested by Dr. Fisher.

REFERENCES

1. *Column Research Council* Guide to Design Criteria for Metal Compression Members, *1960.*
2. *Timoshenko, S. and Gere, M. N.* Theory of Elastic Stability, *McGraw-Hill, New York, 1961.*
3. *Bleich, H.* Buckling Strength of Metal Structures, *McGraw-Hill, New York, 1952.*
4. *Pincus, G.* The Performance of Columns and Beams Continuously Braced with Diaphragms, *Ph.D. Thesis, Cornell University, Ithaca, New York, 1963.*