Practical Application of Energy Methods to Structural Stability Problems

R. SHANKAR NAIR

INTRODUCTION

Simple energy-based formulations can provide convenient solutions for, and fresh insights into, many common structural stability problems. This is true even today, when practicing engineers have ready access to sophisticated computer programs for first- and second-order structural analysis.

In general, the direct energy-based formulation of a stability problem involves balancing the strain energy resulting from deformation of the structure with the work done by applied loads. The critical load corresponding to any assumed buckled shape can be determined in this manner. In many practical situations, this is a surprisingly simple and straightforward process.

At the most basic level, energy concepts can be used to assess whether a particular deformation pattern in a structure is a potential buckling mode: If the deformation causes the applied loads to move in the direction in which they are applied, i.e., if the loads do positive work, the deformation is a potential buckling mode, otherwise it isn't, regardless of the magnitude of loading. This simple concept can be used to determine whether and where bracing is needed in a structure.

This paper will begin with a brief reintroduction to the energy-based formulation of buckling problems. The energy formulation will be applied to certain structure types for which buckling solutions, by other means, are readily available. This will show how the energy approach can provide fresh insight into the behavior of structures, by furnishing simple physical explanations for some of the characteristics of familiar solutions.

A general procedure for stability analysis based on energy concepts will then be presented, and illustrated with the help of practical examples.

ALTERNATIVE BUCKLING FORMULATIONS

When a structure buckles, the work done by external loads as a result of the buckling deformation must be equal to the internal strain energy developed as a result of that deformation. Thus, if an assumed deformation is a potential buckling mode, the magnitude of applied load at which this equality becomes valid is the corresponding buckling load. This is the basis of the energy formulation of structural buckling problems.

Consider the structure illustrated in Figure 1. A vertical rigid strut of length L is hinged at the bottom and supported laterally at the top by a spring of stiffness k (force/length). A vertical load P is applied at the top. The task is to find the critical value of P at which the strut will buckle laterally at the top.

Figure 1 shows the original, undeformed configuration of the structure and the configuration and forces when the strut is displaced laterally by a distance d at the top. Note that as the strut rotates about the hinge at its base, the horizontal displacement d at the top produces a vertical movement a. The critical value of P, denoted as P_{cr}, will be determined by two alternative methods.

Formulation Based on Statical Equilibrium

For the structure to be in equilibrium in the displaced (buckled) state shown in Figure 1, the moments of forces P and kd about the hinge at the base must be equal and opposite, or $Pd = kdL$. The value of P that satisfies this condition, P_{cr}, is kL.

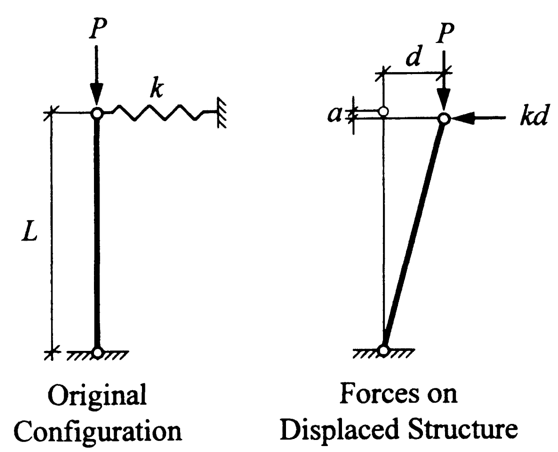

Fig. 1. Buckling of hinged strut restrained by spring.

R. Shankar Nair, Ph.D., P.E., S.E. is senior vice president of Teng & Associates, Inc., Chicago.

Energy-Based Formulation

The strain energy in the compressed spring in the displaced structure is $kd^2/2$. As the top of the strut is displaced laterally, the point of application of load P moves downward by distance a. The work done by P as the structure is displaced is Pa. It can be shown (see Appendix) that for small values of d:

$$a = d^2 / (2L) \tag{1}$$

Substituting for a, the work done by P is $Pd^2/(2L)$. For buckling, this must be equal to the strain energy, $kd^2/2$. The buckling load, which is the value of P that satisfies the energy equality, is given by:

$$P_{cr} = kL \tag{2}$$

This is, of course, the same result as was obtained by the direct statical equilibrium approach.

Formulation Based on Response to Lateral Load

In typical structural engineering applications, the structure stiffness (k in Figure 1) is not known explicitly. What is known is the response of the structure to lateral load, determined from a linear small-deformation analysis. To simulate this situation in the above example, consider that k is not known, but it is known that the structure undergoes lateral displacement d when loaded with a lateral force H. The critical vertical load, P_{cr}, could, of course, be determined by substituting H/d for k in Equation 2. Alternatively, a derivation that does not involve stiffness at all could be used, as follows.

The work done by horizontal load H in inducing horizontal displacement d is $Hd/2$. Therefore, the strain energy corresponding to displacement d (regardless of whether it was caused by direct lateral loading or by buckling due to vertical load) must also be $Hd/2$. The work done by load P as the structure buckles laterally was calculated previously as $Pd^2/(2L)$. Equating $Pd^2/(2L)$ and $Hd/2$, the following equation for buckling load is obtained:

$$P_{cr} = HL / d \tag{3}$$

Equation 3 is applicable, either as-is or with minor alterations, to a whole range of common structures, where P_{cr} is the critical vertical load for lateral buckling, L is the height over which the lateral displacement occurs (the "story" height, in many typical applications), and d is the lateral displacement caused by lateral load H (acting alone, without simultaneous vertical loading). This is the basis for one of the stability formulas (for B_2 in LRFD Equation C1-4) in the AISC LRFD Specification.[1]

STUDY OF ONE-BAY FRAME

A simple, symmetrical, one-bay portal frame will be studied next. Two forms of this frame will be studied: first, one in which the columns remain straight as the structure deforms laterally; then, one in which the beam remains straight and the columns bend. A frame of the first type is shown in Figure 2; a frame of the second type in Figure 3.

The frame in Figure 2 has rigid columns and a flexible beam. Lateral load H produces lateral displacement d in this frame. Consider lateral buckling of this structure under total vertical load P applied equally at the upper corners. When the frame buckles laterally by a distance d, the points of application of vertical load move downward by a distance a. The work done by the vertical loads is Pa or, substituting from Equation 1, which is valid for this frame, $Pd^2/(2L)$. Since lateral load H produces lateral displacement d, the strain energy corresponding to lateral displacement d is $Hd/2$. Equating $Pd^2/(2L)$ and $Hd/2$, the critical load for buckling, P_{cr}, is HL/d, as in Equation 3.

The frame in Figure 3 is similar to that in Figure 2, except that the beam is rigid and the columns flexible. The lateral stiffness of the frame, defined by load H required to produce displacement d, is the same in both structures. In the derivation of the critical load for buckling, the difference between the structures is in the relationship between the lateral displacement, d, and the corresponding vertical movement, a, at the top of the columns. Equation 1, relating a and d, is valid

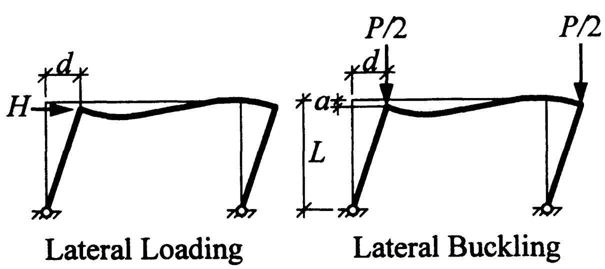

Fig. 2. One-bay frame in which the columns remain straight.

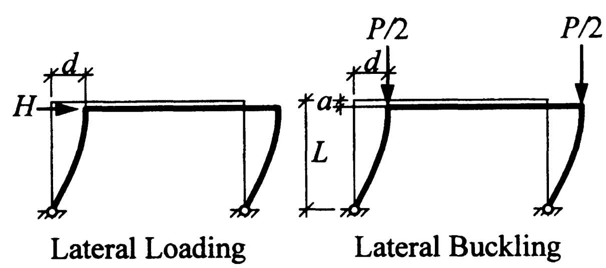

Fig. 3. One-bay frame in which lateral deformation causes column bending.

only when the member rotates about one end but remains straight, as in the structures in Figures 1 and 2. When the columns bend as in the frame in Figure 3, the approximate relationship between a and d is (see Appendix):

$$a = 1.2d^2/(2L) \qquad (4)$$

Equating Pa with $Hd/2$ and substituting from Equation 4 yields the critical vertical load for lateral buckling:

$$P_{cr} = 0.82HL/d \qquad (5)$$

Comparing Equation 5 with Equation 3 (which is applicable to the structure in Figure 2), it is apparent that even though the structures in Figures 2 and 3 have the same lateral stiffness (defined as the lateral load required to produce a particular lateral displacement), the structure in Figure 3 buckles laterally at a lower vertical load. The reason for this difference is easy to explain using energy principles, as follows.

For a given lateral displacement, the bending of the columns in the Figure 3 structure causes additional downward movement at the top, compared to that in the structure in which the columns remain straight. (Compare Equations 1 and 4.) The vertical loads thus move downward through a greater distance; so they do more work for a given magnitude of load. Since lateral buckling occurs when the work done by the vertical loads is equal to the strain energy created by lateral displacement (same in both cases), the critical load is lower for the structure in which the columns bend and the vertical loads move through a greater distance.

Note on Accuracy

It should be noted that while Equation 3 is exact for the structures in which the columns remain straight, Equation 5 is approximate (though sufficiently accurate for design-office purposes) for the structure shown in Figure 3. The reason is, first, that Equation 4 may be in error by a small amount. More importantly, the shape of the deformed columns may not be exactly the same in the laterally loaded frame (left side of Figure 3) as in the laterally buckled frame (right side of Figure 3). As a result, the strain energy in the columns corresponding to lateral displacement d may not be exactly the same in the laterally buckled structure as in the laterally loaded structure.

FRAME WITH MULTIPLE COLUMNS

As an extension of the study of two idealized types of single-bay frames, similar analysis procedures will now be applied to a story in a low-rise building with several columns. The building is assumed to be one in which lateral displacements caused by lateral loading are the result of shear racking; lateral displacements due to axial length changes in columns are considered to be negligible. (This type of behavior is typical of low-rise rigid-frame buildings.)

The building story to be studied is shown in Figure 4. It is meant to represent a general structure with a wide range of column-beam stiffness ratios and restraint conditions: Column 2 (second column from the left) is pin-connected, while all others are rigidly connected to the beams at the top and bottom of the story; Column 5 is extremely stiff in comparison with the beams connected to it; Column 3 is extremely flexible in comparison with the beams. The story height is L. Lateral load analysis indicates that lateral shear force H produces lateral deformation d in this story, as shown in Figure 4.

The vertical loads on the five columns are denoted as P_1, P_2, ... P_5. The laterally buckled shape of the structure (scaled or normalized to have a story displacement of d) is shown in Figure 5. The vertical shortening of the story as it is deformed laterally by d is a_1, a_2, ... a_5 at Columns 1, 2, ... 5. This vertical shortening is caused by the rotation and bending of the columns as the story is displaced laterally, not by axial length reduction due to compression (which should not change significantly as a result of the lateral displacement).

When the structure is loaded laterally as indicated in Figure 4, the work done by lateral force H as the story is deformed through distance d is $Hd/2$. Therefore, the strain energy in the structure corresponding to the deformed shape shown in Figure 4 must be $Hd/2$. The shape of the laterally buckled structure (Figure 5) is approximately the same as that of the laterally loaded structure (Figure 4). Therefore the strain energy in the buckled structure must also be $Hd/2$.

For buckling to occur, the total work done by vertical loads P_i as they move through vertical distances a_i must be

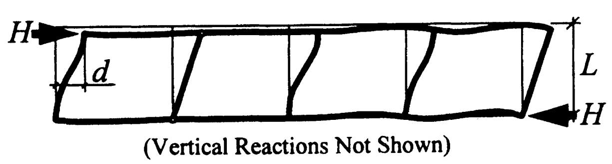

Fig. 4. Lateral loading of story of a building.

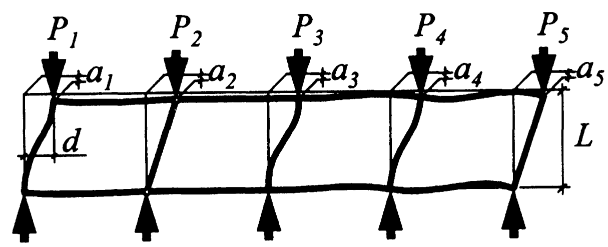

Fig. 5. Lateral buckling of story of a building.

equal to the strain energy. So, denoting the critical values of the vertical load as P_{1cr}, P_{2cr}, ... P_{5cr}, the following is obtained:

$$\Sigma P_{icr}\, a_i = Hd/2 \qquad (6)$$

Though the buckled shape is assumed to be the same as the laterally loaded shape, displacements a_i cannot be obtained directly from the results of the lateral load analysis, unless that analysis was a large-deformation analysis. Typically, the lateral load analysis is a small-deformation analysis and would, therefore, show the downward displacements caused by the lateral bending of columns to be zero. An indirect approach to determining the downward displacements is required.

Displacements a_i can be expressed in terms of the lateral displacement d as follows:

$$a_i = \gamma_i d^2/(2L) \qquad (7)$$

and Equation 6 can be rewritten as:

$$\Sigma P_{icr}\, \gamma_i = HL/d \qquad (8)$$

where γ_i depends on the shape of the column in the buckled structure. At columns that remain essentially straight (Columns 2 and 5 in this example), γ is unity and Equation 7 becomes the same as Equation 1. At columns that bend in reverse curvature with little rotation at top and bottom (Column 3), γ is 1.2, as in Equation 4. Columns that undergo intermediate patterns of deformation have values of γ between 1 and 1.2.

Given the small range of γ values, it should not be difficult for the engineer to estimate appropriate values of γ based on relative member stiffnesses or a knowledge of the shape of the columns in the laterally loaded structure. [It is theoretically possible, in certain very unusual situations (not illustrated in Figures 4 or 5), for γ to be greater than 1.2. The engineer who is aware of the physical meaning of the γ term should be able to identify these situations and estimate an appropriate value.]

Equation 8 can be rewritten to yield the following explicit formula for the total vertical load that would cause lateral buckling of the story:

$$\Sigma P_{icr} = \chi HL/d \qquad (9)$$

where

$$\chi = \Sigma P_i \;/\, \Sigma(P_i\, \gamma_i) \qquad (10)$$

The usual range of χ is from 0.82 to 1. If most of the vertical load on the story is on columns that remain essentially straight, χ would approach 1; if most of the load is on columns that bend in reverse curvature with little rotation at top and bottom, χ would approach 0.82.

Note on Accuracy

There are two possible sources of error in the formulas for buckling load (Equations 8 and 9). First, errors are possible in the estimation or selection of γ values, which are based on the shapes of the columns in the laterally buckled story. The normal range of γ is from 1 to 1.2. Second, errors can arise from differences between the deformed shapes of the laterally loaded structure (Figure 4) and the laterally buckled structure (Figure 5). The derivation of the buckling formulas assumes that the deformations are essentially similar and, therefore, that they give rise to the same strain energy. If this is not true, errors will result.

GENERAL BUCKLING ANALYSIS PROCEDURE

The following is a general energy-based procedure for the buckling analysis of a structure loaded with a given system of loads:

Step 1

Assume a buckled shape.

Step 2

At each point of load application, determine the displacement caused by the assumed buckling.

Step 3

Calculate the total work done by the applied loads when the structure buckles as assumed. (The work done by each load component is the magnitude of the load times the movement of the point of application of the load, as determined in Step 2, in the direction of the load.)

If the work calculated in Step 3 is zero or negative, the assumed buckled shape is not a potential mode of buckling, regardless of the magnitude of the applied loads. (That is, the given system of loads would not cause buckling corresponding to the assumed shape, even if the loads were increased uniformly in magnitude. However, if the loading has multiple components, changing the relative value of the components may change this conclusion. Steps 2 and 3 should be repeated for the new load pattern.)

If the work calculated in Step 3 is positive, proceed as follows to determine the critical load (as a multiple of the given applied loads) for buckling corresponding to the assumed buckled shape.

Step 4

Select a load or system of loads that will cause deformation of the structure similar to the assumed buckled shape. Analyze the structure under this artificial loading (which is completely unrelated to the actual loading under which the structure is being analyzed for buckling). Stability and "*P*-delta" effects should be neglected in this analysis.

Step 5

Determine the total work done by the "artificial" loads applied in Step 4. (The work done by each load component is the magnitude of the load times the distance through which the point of application of the load moves in the direction of the load, divided by two.)

The strain energy in the structure under the loads applied in Step 4 is equal to the work done by the loads (calculated in Step 5). Since the shape of the buckled structure is the same, it should involve the same strain energy. Therefore, the strain energy in the buckled structure also is equal to the work calculated in Step 5.

Step 6

Check for buckling. The structure will buckle as assumed in Step 1 if the work calculated in Step 5 is less than that calculated in Step 3. The multiple of the actual loads that will cause buckling corresponding to the assumed buckled shape is the work calculated in Step 5 divided by the work calculated in Step 3.

Note on Accuracy

The critical load determined by this procedure represents an upper-bound solution; a different assumed buckled shape (Step 1) may indicate a lower critical load. For the assumed buckled shape, the accuracy of the result depends on the accuracy of Step 2 (calculation or estimation of displacements at load points) and on the similarity between the buckled shape and the shape produced by the "artificial" loads applied in Step 4. The engineer who understands the physical meaning of the possible sources of error in the analysis should be able to make an informed judgment as to the likely accuracy of the result.

EXAMPLES OF APPLICATION OF GENERAL PROCEDURE

In the following, the use of the suggested general energy-based buckling analysis procedure will be demonstrated by applying it to specific problems.

Frame with X Bracing

A one-story, one-bay, X-braced frame is shown in Figure 6. The frame is 15-ft high and 25-ft wide (center-to-center dimensions). The columns are W10×33; the beam at the top is W16×26. The bracing diagonals are 4×4×3⁄8 tubes. The beam-to-column connections and column bases are idealized as being pinned. The beam and the columns are restrained perpendicular to the plane of the frame.

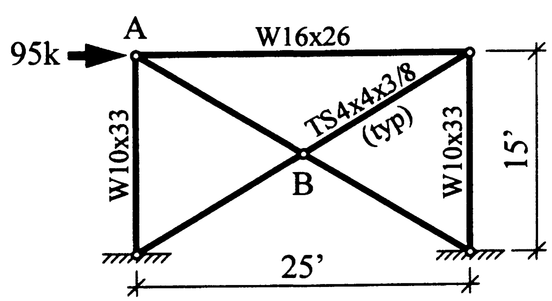

Fig. 6. Frame with X bracing.

The bracing members are pinned in both directions at the ends and where they intersect at the center. (Alternative connection conditions at the center will be considered subsequently.)

The loading is 95 kips applied horizontally at Point A at the top, as shown. Linear, planar frame analysis shows that the resulting forces in the diagonals are 64 kips in the compression diagonal and 46 kips in the tension diagonal. The compression that would cause buckling of a diagonal between the end and the center (a TS4×4×3⁄8 member with a length of 175 inches, pinned at both ends) is 100 kips. This corresponds to an applied load of 147 kips at Point A.

The task at hand is to determine, first, whether out-of-plane displacement of the point of intersection of the diagonals, Point B, is a potential buckling mode. (Does the tension diagonal brace the compression diagonal at this point?) If out-of-plane displacement of Point B is a potential buckling mode, what is the stiffness of the external restraint required at Point B to prevent this form of buckling?

The analysis that follows is keyed to the steps in the proposed general buckling analysis procedure.

Step 1

The buckling mode to be tested is a 1-in. out-of-plane displacement of Point B.

Step 2

Now we must find the movement of Point A caused by a 1-in. out-of-plane displacement of Point B. This could be done by three-dimensional large-deformation analysis of the structure. It could also be done (far more practically) using simple, planar, linear, small-deformation analysis, as follows.

The effect on the planar structure of the 1-in. out-of-plane movement of Point B can be simulated by a shortening of the four brace members. The amount of the shortening is as indicated by Equation 1. The length of each member is 175 inches, and the shortening is $(1)^2/(2 \times 175)$, or 0.00286 inch.

Input of the brace member length reductions of 0.00286 inch (simulated as a temperature drop of 2.51°F) into a planar frame analysis computer program indicated the structure deformations shown in Figure 7. Point A moves down by 0.00062 inch and to the right by 0.00108 inch.

Step 3

The work done by the applied load of 95 kips when Point B is displaced out-of-plane by 1 inch is 95 kips times 0.00108 inch, or 0.103 kip-in.

Since this is a positive amount of work, out-of-plane displacement of Point B *is* a potential buckling mode for this structure. The tension diagonal *does not* brace the compression diagonal, at least not sufficiently to eliminate this mode of buckling.

Note that if the beam at the top of the frame were of infinite axial stiffness, Point A would not move horizontally and the applied load would do no work. Also, if the 95-kip load were split into two 47.5-kip loads, one at each corner of the frame, one load would do positive work and the other would do negative work, for a total of zero. Finally, if the 95-kip load were applied at the opposite end of the top beam, the work done would be negative. In all of these cases, out-of-plane displacement of Point B would not be a potential buckling mode.

Step 4

Having established that the displacement being considered is a potential buckling mode, the next step is to select an "artificial" loading that will produce the assumed 1-in. out-of-plane displacement of Point B. In the fully pin-connected structure, the out-of-plane displacement could be produced by an infinitesimally small out-of-plane load applied at Point B.

Step 5

The work done by the infinitesimally small load applied in Step 4 as it moves through 1 inch to produce the 1-in. displacement would also be infinitesimally small. This is the strain energy corresponding to the assumed buckled shape.

Step 6

The strain energy corresponding to the assumed buckled shape is smaller by an infinite amount than the 0.103 kip-in. of work done by the 95-kip applied load, as calculated in Step 3. [The beam and column length changes indicated in Figure 7 do imply strain energy in these members. However, this strain energy is of a lower order of magnitude than the work done by the applied load.] This indicates that any load at all applied at Point A, acting toward the right, would cause out-of-plane buckling of Point B. That is, the critical load for this buckling mode is zero.

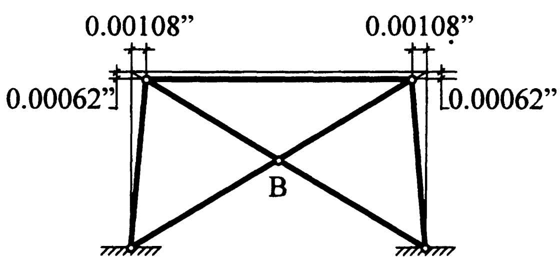

Fig. 7. Deformation due to 1-in. out-of-plane displacement of point B.

Structure with External Restraint at Point B

Consider the same structure, but with an external out-of-plane restraint at Point B of stiffness *k* kips/inch. What does *k* have to be to prevent buckling under the specified 95-kip load?

Going back to Step 4, the loading that will produce a 1-in. out-of-plane displacement of Point B in this new structure is a force of *k* kips applied at Point B. The work done by this force as it produces the 1-in. displacement is *k*/2 kip-in. (Step 5). This is the strain energy. Equating this with the 0.103 kip-in. of work done by the applied load, the required value of *k* to prevent buckling under the 95-kip applied load is found to be 0.206 kips/in.

Diagonals Continuous Across Intersection

Consider, next, how the behavior of the frame changes when both the bracing diagonals are continuous, in the out-of-plane direction, across the intersection at Point B (with no external restraint).

Since the diagonals now bend into a curved form when displaced out-of-plane, the effective shortening of the braces corresponding to the 1-in. out-of-plane displacement of Point B is better represented by Equation 4 than by Equation 1, and works out to 0.00343 inch. The horizontal movement of Point A increases from 0.00108 inch to 0.00130 inch, and the work done by the 95-kip load becomes 0.124 kip-in.

The out-of-plane force required at Point B to produce a 1-in. displacement at that location (by bending the two diagonals, each acting as a beam spanning 350 inches, by 1 inch at the center) is 0.694 kip. The work done by this force, 0.347 kip-in., is the strain energy. This is more than the work done by the 95-kip applied load. So the structure will not buckle at the specified loading.

The critical load for out-of-plane buckling of the diagonals at the center is 95(0.347/0.124), which amounts to 266 kips. Of course, since the critical load for buckling of the individual compression half-diagonals is much lower (147 kips), the structure will not reach the load that would cause out-of-plane buckling at Point B. The compression diagonal is, in effect, adequately braced by the tension diagonal.

One Diagonal Continuous Across Intersection

When the compression diagonal alone is continuous across the point of intersection, the critical load was found to be 93 kips, which is slightly less than the specified applied load and much less than the load that would cause buckling of the individual compression half-diagonals. This indicates that the tension diagonal *does not* fully brace the compression diagonal.

When the tension diagonal alone is continuous, the critical load is higher than the 147 kips that would cause buckling of the individual compression half-diagonals. The tension diagonal *does* fully brace the compression diagonal in this case.

Truss Between Columns

The next example is a deep, light truss between two columns, shown in Figure 8. The truss spans 48 feet and is 8-ft deep. It has six panels of 8 feet each. The columns extend from the top of the truss to 16 feet below the bottom chord. The columns are W14×145 sections, oriented with their webs in the plane of the truss. The upper and lower chords are WT7×24 and WT7×17 sections, respectively. All truss verticals and diagonals are 3×3×3/8 double angles. The columns are fixed at their bases. The columns and upper chord are restrained perpendicular to the plane of the truss.

The applied loading is a vertical force of 20 kips at each upper-chord panel point. The member forces that result from this loading are indicated in Figure 8 (in kips, tension positive, chord forces shown in one half, vertical and diagonal member forces in the other half). It may be noted that the outermost lower chord members, L_1L_2 and L_6L_7, are in compression.

Assuming all truss connections to be pinned in both directions (a conservative simplifying assumption), and neglecting initial out-of-straightness and other imperfections, does the lower chord require bracing at L_2 and L_6? If bracing is required, how stiff does it have to be for the structure to resist the specified loading without lateral buckling of the lower chord?

The buckling displacement to be considered is a 1-in. lateral (out-of-plane) movement of the lower chord from L_2 to L_6. The effect of this out-of-plane deformation on the planar structure can be simulated by a shortening of the members that undergo out-of-plane rotation. Using Equation 1, the magnitude of the effective length changes is found to be as follows:

$1/(2 \times 96) = 0.005208$ inch for L_1L_2 and L_6L_7

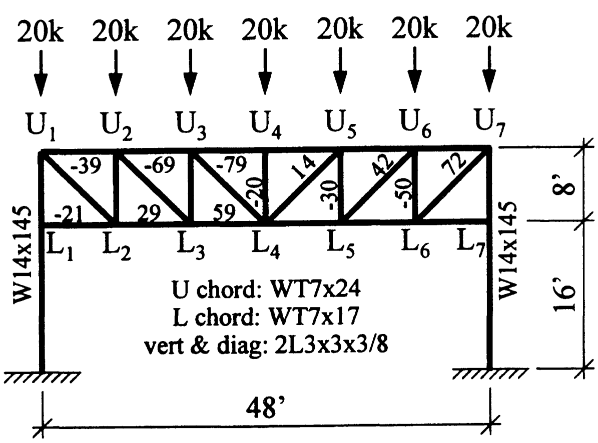

Fig. 8. Truss between two columns.

$1/(2 \times 136) = 0.003683$ inch for all truss diagonals

$1/(2 \times 96) = 0.005208$ inch for all truss verticals

The structure was analyzed for these member length changes using a planar frame analysis computer program. The length changes were simulated by temperature drops of 8.34°F in L_1L_2, L_6L_7, and the truss verticals; 4.17°F in the truss diagonals. The resulting vertical movement of the top of the truss is indicated in Figure 9.

Since the entire upper chord moves downward, it is clear, without further calculation, that the applied loads do positive work when the lower chord between L_2 and L_6 is displaced laterally. This form of lateral displacement does, therefore, represent a buckling mode. Lateral restraint is required to prevent this type of buckling.

The next task is to determine the required stiffness, k (kips/in.), of the lateral restraints at L_2 and L_6. The total work done by the two restraints as the chord is displaced laterally by 1 inch is $2k/2$, which is k kip-in. This is the strain energy corresponding to the assumed buckling mode. The work done by the applied loads as the upper chord moves down as indicated in Figure 9 is

$$20(0.00157) + 20(0.00245) + 20(0.00264) + 20(0.00245) + 20(0.00157)$$

which amounts to 0.2136 kip-in. Equating the two work quantities, the required stiffness of each restraint is 0.2136 kip/in.

With the chord restrained laterally at L_2 and L_6, is lateral displacement of L_3, L_4 and L_5 a potential buckling mode? To check this possibility, the vertical movement of the top of the truss due to a 1-in. out-of-plane displacement of L_3, L_4 and L_5 was determined.

As shown in Figure 10, lateral displacement of the lower chord between L_3 and L_5 results in upward movement of the top of the truss over its entire length, causing the applied loads to do negative work. This indicates that lateral displacement of L_3, L_4 and L_5, while L_2 and L_6 are laterally restrained, is not a potential buckling mode.

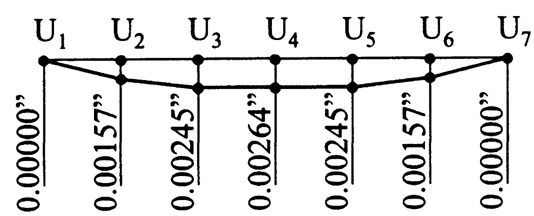

Fig. 9. Vertical displacements at top of truss due to 1-in. lateral displacement of lower chord between L_2 and L_6.

King Post Truss

As a final example, consider the king-post truss shown in Figure 11. The upper chord is restrained laterally. Is lateral (out-of-plane) displacement of the bottom of the post a potential buckling mode?

This buckling problem, which has been the subject of much debate and confusion among engineers, is not really a buckling problem at all. It is simply an issue of statical equilibrium (with one possible exception, as will be explained). This becomes obvious when the truss is imagined as being encased within a rigid, solid body, supported as the truss is supported, as shown in Figure 12.

It should be obvious from Figure 12 that the rigid body will be in neutral equilibrium if the load is applied precisely on the straight axis between the two supports. The body will rotate about the axis if the load is offset laterally from the axis. It will be in stable equilibrium if the load is applied directly below the axis. It will be in unstable equilibrium if the load is applied directly above the axis. (It is only the last situation that could be regarded as a "stability" problem.)

What this means in practice, in the design of a real king-post truss, is that unless the point of application of vertical load is definitely and reliably known to lie directly below the potential axis of rotation of the truss (which is not easy to identify in a real structure), the post must be restrained against out-of-plane rotation. This could be done through lateral restraint at the lower end or rotational restraint at the top.

If the load may be offset laterally from the axis, the required capacity of the restraint could be determined through statics. If the load may be offset vertically, the energy-based techniques presented in this paper could be used to determine the required stiffness of the restraint, as follows.

Consider a king-post truss of the configuration shown in Figure 11, with an 8-ft long post and a load of 10 kips applied 10 inches above the axis of out-of-plane rotation of the truss. A 1-in. lateral displacement of the lower end of the post causes an 0.104-in. lateral movement of the point of application of the load. The resulting vertical movement of the load is $(0.104)^2/(2 \times 10)$, which amounts to 0.00054 inch, and the work done by the load is 0.0054 kip-in. To balance this work, the stiffness of a lateral restraint at the lower end of the post must be 0.0108 kip/in. (Alternatively, a rotational restraint of equivalent stiffness could be used at the top.)

SUMMARY AND CONCLUSIONS

A brief reintroduction to the energy-based formulation of buckling problems has been presented, and a practical, step-by-step procedure for the use of energy-based buckling analysis in the design office has been suggested. The proposed procedure has been demonstrated with the help of practical examples.

It has been shown that many structural stability problems can be solved economically and conveniently using the suggested energy-based techniques. Just as importantly, the energy formulation of buckling problems can give the practicing engineer fresh insight into the behavior of structures.

REFERENCE

1. American Institute of Steel Construction, *Load and Resistance Factor Specification for Structural Steel Buildings*, Chicago, IL, 1993.

APPENDIX—DERIVATION OF GEOMETRIC RELATIONSHIPS

In this Appendix, formulas for the length differences a defined in Figure 13 will be derived. The offset d is very small in comparison with L.

Straight Offset

From first principles: $(L - a)^2 = L^2d^2$

Expand to: $L^2 - 2aL + a^2 = L^2d^2$

Neglect a^2 since it is of a lower order of magnitude than the other terms. Upon simplifying:

$$a = d^2 / (2L)$$

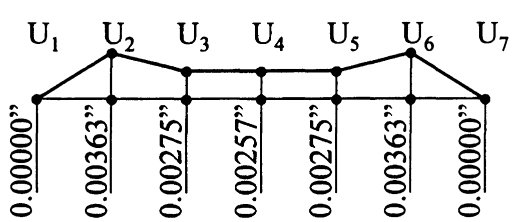

Fig. 10. Vertical displacements at top of truss due to 1-in. lateral displacement of lower chord between L_3 and L_5.

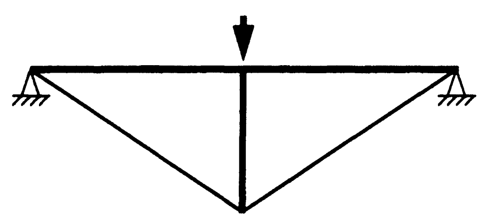

Fig. 11. King-Post truss.

Curved Offset

Consider a sinusoidal curve. Denoting $L\text{-}a$ as l, the curve is defined by:

$$y = d \sin \pi x / (2l)$$

The length of the curve is given by (with all integration over l):

$$L = \int [1 + (dy/dx)^2]^{1/2} dx$$

$$= \int [1 + \{\pi^2 d^2 / (4l^2)\} \cos^2 \{\pi x / (2l)\}]^{1/2} dx$$

For small values of ε, $[1 + \varepsilon]^{1/2}$ approaches $[1 + \varepsilon / 2]$. Using this in the equation for the length of the curve (again with all integration over l):

$$L = \int [1 + \{\pi^2 d^2 / (8l^2)\} \cos^2 \{\pi x / (2l)\}] dx$$

$$= l + \{\pi^2 d^2 / (8l^2)\} \int \cos^2 \{\pi x / (2l)\} dx$$

Using the known definite integral solution, $\int^{\pi} \cos^2 z . dz = \pi / 2$, the integral above is found to be $l/2$ and the following is obtained by substituting and simplifying:

$$a = L - l$$

$$= \{\pi^2 d^2 / (8l^2)\}[l/2]$$

$$= (\pi^2 / 16) d^2 / l$$

$$= 1.23 d^2 / (2l)$$

Neglecting lower-order differences, this can be written as:

$$a = 1.23 d^2 / (2L)$$

This is the solution for a sinusoidal curve (which is the buckling mode of an axially-loaded prismatic bar). Other plausible curve shapes will yield slightly different results. A coefficient of 1.2 (instead of 1.23) is used in the main text of the paper.

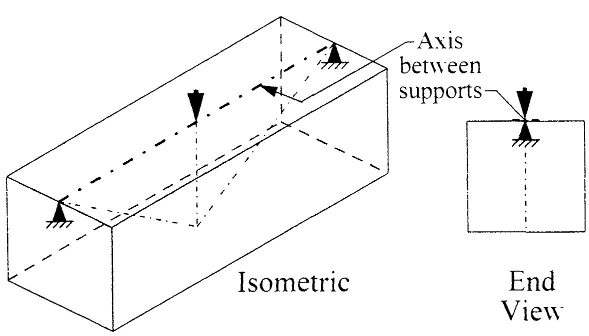

Fig. 12. King-Post truss encased within rigid body.

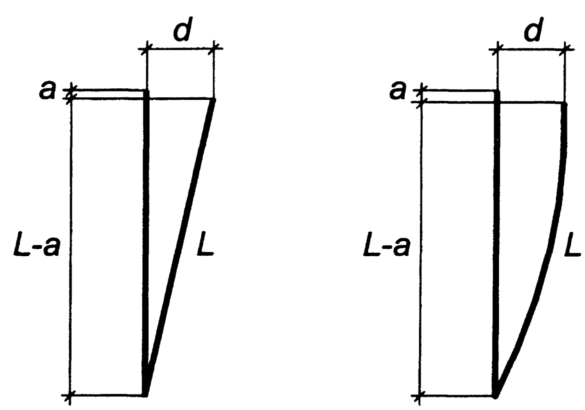

Fig. 13. Definition of geometric relationships.