Technical Note

Internal Second-Order Stiffness: A Refined Approach to the R_M Coefficient to Account for the Influence of P-δ on P-Δ

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ABSTRACT

One component of the B_2 amplifier method of addressing second-order effects is the R_M coefficient, which represents the influence of P-δ on P-Δ effects. This paper presents the background for R_M based on LeMessurier's paper, "A Practical Method of Second-Order Analysis: Part 2—Rigid Frames," (1977), and makes explicit the simplifications entailed in the AISC *Specification for Structural Steel Buildings* (AISC, 2016b) formulation for this coefficient. These simplifications, while providing for reliable strength design, can overestimate the P-δ effect for typical building applications, especially if applied to drift. A simple formula for R_M based on the work of LeMessurier permits a more precise estimate, which can be used as a component of both force and displacement amplifiers presented in this paper. This explicit approach to the R_M coefficient provides the basis for clear presentation of the relationships first-order and second-order stiffness (both internal and external), including the distinct effects of P-δ and P-Δ stiffness reductions on equilibrium at the second-order displacement.

Keywords: second-order analysis, stability.

INTRODUCTION

The AISC *Specification for Structural Steel Buildings* (AISC, 2016b), hereafter referred to as the AISC *Specification*, presents an approximate method for second-order analysis in Appendix 8, using a factor B_2 to amplify forces to account for P-δ and P-Δ effects. While the application of this method is clear in the *Specification*, the derivation is not presented, and the implicit simplifications made can easily be missed. This paper traces the connections between the source material [the landmark LeMessurier paper, "A Practical Method of Second-Order Analysis: Part 2—Rigid Frames," (1977)], and the method in the *Specification*. In the process, the relationship between the distinct amplifiers for force (B_2) and displacement (presented here as D_{AF}) is made explicit, and a refined formulation of R_M is presented.

The use of the refined formulation of R_M permits minor reduction in conservatism for typical building cases. Additionally, the availability of more accurate hand methods of calculating second-order effects empowers engineers to better understand and critically evaluate the results of computerized second-order analyses. The reconciliation of the B_2 amplifier with amplified displacements may be of use to engineers employing the amplifier method. Understanding the basis of the amplifier method is particularly helpful to students, as is understanding the relationships between first-order stiffness and internal and external second-order stiffness.

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A REFINED R_M

Force Amplification

The amplifier-based method of second-order analysis, as presented in AISC *Specification* Appendix 8 (2016b), requires consideration of the influence of P-δ on P-Δ effects. This is represented by the coefficient R_M, which is incorporated into the AISC *Specification* equation for the force amplifier B_2. The B_2 amplifier is defined by *Specification* Equations A-8-6 and A-8-7, which can be combined and expressed as:

$$B_2 = \frac{1}{1-\dfrac{P_{story}\Delta_1}{R_M HL}} \tag{1}$$

where

B_2 = force amplification factor for second-order effect

H = first-order shear, kips (N)

L = story height, in. (mm)

P_{story} = total gravity load, $P_{mf} + P_{lean}$, at LRFD level, kip (N)

P_{lean} = gravity load on non-moment-frame columns, kip (N)

P_{mf} = gravity load on moment-frame columns, kip (N)

R_M = stiffness-reduction coefficient to account for member P-δ influence on structure P-Δ

Δ_1 = first-order story drift corresponding to load H (Δ_H in the AISC *Specification*), in. (mm)

The AISC *Specification* LRFD/ASD adjustment factor α is omitted from the gravity-load definitions for brevity.

The R_M term in Equation 1 effectively reduces the lateral stiffness of the system from the first-order stiffness (H/Δ_1), and the reduced stiffness with P-δ included ($R_M H/\Delta_1$) is used to determine the P-Δ effect. (Note that in this study, the terms B_2 and R_M represent their functions as described in their definitions, rather than the formulas for these quantities in the AISC *Specification*. More accurate formulas for these quantities are presented later.)

The B_2 amplifier can be used to determine the overturning moment corresponding to equilibrium in the deformed condition (i.e., at the second-order drift Δ_2), as shown for a simple structure in Figure 1:

$$B_2 HL = HL + P_{story}\Delta_2 \tag{2}$$

where

Δ_2 = second-order story drift, in. (mm)

(This amplification also applies to the shear at the sloped top of the cantilever column, considering the vertical force P_{mf}.)

The B_2 amplifier can be expressed as a function of the second-order story drift Δ_2 by rearranging Equation 2:

$$B_2 = 1 + \frac{P_{story}\Delta_2}{HL} \tag{3}$$

The symbol F_{AF} can denote a force amplification factor ["FAF" in Equation A-6 in Griffis and White (2013)], which by definition is equal to B_2:

$$F_{AF} = B_2 \tag{4}$$

Displacement Amplification

A displacement amplification factor, D_{AF} ("DAF" in Equation A-7 in Griffis and White [2013]), can be defined as:

$$D_{AF} = \frac{\Delta_2}{\Delta_1} \tag{5}$$

Combining Equations 1, 3, and 5, the displacement amplifier D_{AF} can be related to B_2 and R_M:

$$D_{AF} = \frac{B_2}{R_M} \tag{6}$$

The R_M coefficient can be defined as the ratio of force to displacement amplification by rearranging Equation 6:

$$R_M = \frac{B_2}{D_{AF}} \tag{7}$$

$$R_M = \frac{B_2\Delta_1}{\Delta_2} \tag{8}$$

The accuracy of both the forces obtained with the amplifier B_2 in Equation 1 and the drifts computed using Equation 6 are dependent on the accuracy of the coefficient R_M. The AISC *Specification* (2016b) provides a conservative, approximate formula for R_M (*Specification* Equation A8-8):

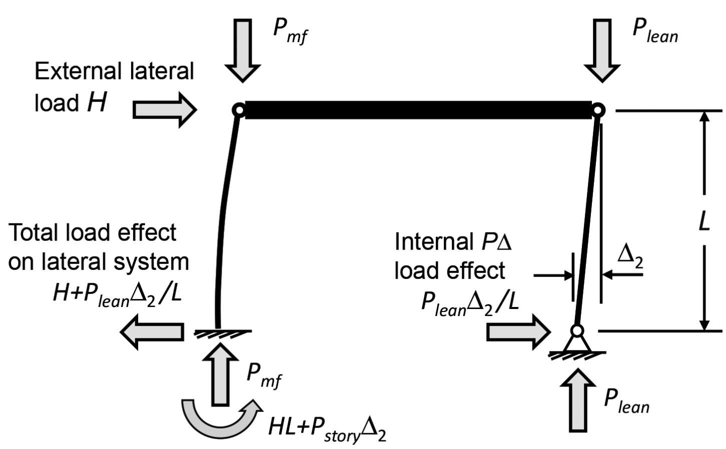

Fig. 1. Equilibrium diagram at second-order displacement.

$$R_M = 1 - 0.15\frac{P_{mf}}{P_{story}} \tag{9}$$

The AISC *Specification* Commentary explains that the minimum value of 0.85 (when P_{mf}/P_{story} = 1.0) represents a lower bound based on LeMessurier's work. While Equation 9 is suitably conservative for reliability in strength design, it can overestimate the P-δ effect on drift if used in Equation 6, especially for stiff systems, and a more precise expression of R_M may be obtained based on LeMessurier's work.

Column Flexural Stiffness Reduction Due to *P*-δ Effects

Equation 37 in LeMessurier's paper presents the force-amplification factor (A.F.), which has the same function as B_2. Changing terms in LeMessurier's equation to be consistent with those used earlier gives:

$$B_2 = \frac{1}{1 - \dfrac{P_{story}}{\dfrac{HL}{\Delta_1} - C_L P_{mf}}} \tag{10}$$

where

C_L = flexural stiffness-reduction coefficient for a moment-frame column due to P-δ effects

The coefficient C_L is potentially different for each column, varying with its deflected shape. For cantilever columns and for moment frames with points of inflection near column mid-height and beam mid-length, LeMessurier's Equations 60 and 58, respectively provide an approximation of C_L:

$$C_L \cong \frac{\dfrac{12}{\pi^2} - 1}{(1+G)^2} \tag{11}$$

where

$$G = \frac{\sum \dfrac{EI_{col}}{L_{col}}}{\sum \dfrac{EI_g}{L_g}} \tag{12}$$

and

E = modulus of elasticity, ksi (N/mm^2)
G = moment-frame relative flexural-stiffness parameter
I_{col} = column moment of inertia, in.^{4} (mm^4)
I_g = girder (beam) moment of inertia, in. (mm)
L_{col} = column height, in. (mm)
L_g = girder length, in. (mm)

The "–1" term in Equations 11 is the subtraction of the P-Δ effect on moment-frame columns, as this is accounted for in the effect of P_{story} in Equation 10. The largest possible value of C_L corresponds to the buckled shape of a rigid-base cantilever or fixed-fixed column with rigid beams at the top and bottom (for which G = 0):

$$\begin{aligned} C_L &\le \frac{12}{\pi^2} - 1 \\ &\le 0.216 \end{aligned} \tag{13}$$

As is shown later, conservatively taking the maximum value of C_L from Equation 13 for all columns is less conservative than Equation 9. For real buildings, however, beams framing into columns do not provide rigid restraint against column rotation, and Equation 11 facilitates determination of significantly lower values of C_L than the maximum value from Equation 13. It is possible to establish a maximum value of C_L based on a minimum value for the ratio G (Equation 12) corresponding to typical beam-column proportioning and frame dimensions, further reducing the calculated P-δ effects. (This beam flexibility must also be incorporated into the analysis. The total effect of beam flexibility is, of course, an increase in displacement and a decrease in strength.) For example, Cheong-Siat-Moy (1976) proposed optimal proportioning of frames for drift control, which corresponds to G = 1.0. Similarly, the AISC *Seismic Provisions* (2016a) contain proportioning requirements for beam and column strength for special moment frames that have some correlation to Equation 12. No minimum values for the ratio G are proposed here; engineers may wish to establish such limiting values on a project basis.

LeMessurier presents an equation for a second-order drift ratio (rather than for a displacement amplifier) in Equation 31. In the terms used here, that equation is:

$$\frac{\Delta_2}{L} = \frac{H}{\dfrac{HL}{\Delta_1} - P_{story} - C_L P_{mf}} \tag{14}$$

Following Griffis and White (2013), the displacement amplifier can be determined by combining Equations 5 and 14:

$$D_{AF} = \frac{1}{1 - \dfrac{\Delta_1 \left(P_{story} + C_L P_{mf}\right)}{HL}} \tag{15}$$

When P_{mf} is zero (such as for a building without moment frames or with negligible axial load on the moment-frame columns), R_M is 1.0, and both amplifiers (F_{AF} and D_{AF}) simplify and become equal. Thus, the amplifier B_2 represents both force and displacement amplification if R_M = 1.0:

$$B_2 = \frac{1}{1 - \dfrac{P_{story}\Delta_1}{HL}} \text{ if } R_M = 1.0 \tag{16}$$

The Refined R_M Approach

Combining Equations 7, 10, and 15 gives an expression for R_M consistent with the amplifiers based on LeMessurier:

$$R_M = 1 - \frac{P_{mf}\Delta_1}{HL} C_L \tag{17}$$

Thus, the system stiffness-reduction effect of P-δ on P-Δ represented by R_M is a function of the first-order lateral stiffness (H/Δ_1). For systems with high lateral stiffness, the P-δ effect on P-Δ may be quite small.

Stability-Coefficient-Based Second-Order Amplifiers

The refined formulation of R_M (Equation 17) can be presented as a function of the stability coefficient θ [defined in ASCE/SEI 7 (2016)], which is often used as a measure of second-order effects. The ASCE 7 equation can be presented in terms consistent with those in this paper:

$$\theta = \frac{P_{story}\Delta_1}{HL} \tag{18}$$

where

θ = stability coefficient per ASCE/SEI 7, Section 12.8.7, Equation 12.8-16

Thus

$$R_M = 1 - \theta\frac{C_L P_{mf}}{P_{story}} \tag{19}$$

Incorporating Equation 18, Equations 10 and 15 can be expressed as functions of θ:

$$B_2 = 1 + \frac{1}{\dfrac{1}{\theta} - \left(1 + \dfrac{C_L P_{mf}}{P_{story}}\right)} \tag{20}$$

$$D_{AF} = \frac{1}{1 - \theta\left(1 + \dfrac{C_L P_{mf}}{P_{story}}\right)} \tag{21}$$

Table 1 presents values of force amplifiers (B_2), displacement amplifiers (D_{AF}), and coefficients R_M for a range of stability coefficient θ. Force amplifiers B_2 and coefficients R_M are computed using both AISC *Specification* equations (Equations 1 and 9) and the explicit equations presented in this paper (Equations 20 and 17) based on P_{mf}/P_{story} = 0.333 and 1.0. The displacement amplifiers (D_{AF}) are also based on those values of P_{mf}/P_{story} and use Equation 21. For comparison, values of the amplifier B_2 for $P_{mf} = 0$ (Equation 16) are also presented. Values in Table 1 are calculated utilizing the maximum value of C_L per Equation 13; this simplification allows their general use as the worst case but overestimates the P-δ effect for real buildings with non-rigid beams.

The AISC *Specification* B_2 is conservative for force amplification but is unconservative if used for drift amplification. The range between the explicitly calculated force and displacement amplifiers is very small for $\theta \le 0.25$. The *Specification* B_2 overestimates the force by 5% or less and could be used as an approximate amplifier for drift, underestimating that effect by no more than 2% in that range. Because the explicit values of R_M using Equation 17 are very close to 1.0 for the range of typical building practice ($B_2 \le$ 1.5), the simplification of using B_2 (with the *Specification* value of R_M) as the displacement amplifier will result in less error than using the *Specification* value for R_M in Equation 6. Use of R_M from Equation 17 with Equations 1 and 5 will produce force and drift results closer to an explicit second-order analysis than will use of the *Specification* B_2.

The discrepancy between the AISC *Specification* and explicit amplifiers B_2 increases at larger values of the stability coefficient, and at very large values the explicit force amplifier B_2 (Equation 20) exceeds the *Specification* B_2 (Equation 1) using the *Specification* R_M (Equation 9). The *Specification* R_M matches the explicit R_M (Equation 17) at $\theta = 0.70$, for which B_2 ranges from 3.3 (for $P_{mf}/P_{story} = 0.0$) to 5.7 (for $P_{mf}/P_{story} = 1.0$). Such high second-order effects are not expected in practical building designs. ($B_2 \ge 5.0$, combined with stiffness reduction due to inelasticity, results in instability.) As such, the *Specification* equations produce a reliable upper bound for force amplification on real structures.

According to AISC *Specification* Section C2.1(b), P-δ effects on P-Δ can be ignored for systems meeting certain conditions. Among these conditions are that $P_{mf}/P_{story} \le 1/3$, and $\Delta_2/\Delta_1 \le 1.7$ (using the reduced stiffness of the Direct Analysis Method, which corresponds to 1.5 for a full-stiffness analysis). That range is shaded in Table 1. (The stability coefficient θ corresponds to a first-order analysis with unreduced stiffness.) The values in Table 1 confirm that negligible error in force amplification is expected in that range as a result of using $R_M = 1.0$.

In Table 1, a line is drawn beneath values of $\theta = 0.25$, which is the limit for this coefficient for seismic design in ASCE/SEI 7, Section 12.8.7 (ASCE, 2016). Within that range the smallest value of R_M is 0.95. Thus, for seismic design, the value of R_M could be determined using a value of 0.05 (or lower) in lieu of 0.15 in Equation 9, providing for greater economy. Similar revisions of Equation 9 are possible for other bounds on the design space, such as limiting B_2 or limiting P_{mf}/P_{story}.

SECOND-ORDER STIFFNESS REDUCTIONS

The refined definition or R_M in Equation 17 serves not only as the basis of more accurate force and displacement amplifiers; it also facilitates a more accurate expression of the

Table 1. Amplifiers and RM Values											
	$P_{mf} = 0$	$P_{mf} = 0.333\ P_{story}$					$P_{mf} = 1.00\ P_{story}$				
	Both	AISC *Specification*		Explicit			AISC *Specification*		Explicit		
	B_2	R_M	B_2	R_M	$B_2 = F_{AF}$	D_{AF}	R_M	B_2	R_M	$B_2 = F_{AF}$	D_{AF}
θ	EQ 16	EQ 9	EQ 1	EQ 19	EQ 20	EQ 21	EQ 9	EQ 1	EQ 19	EQ 20	EQ 21
0.05	1.05	0.95	1.06	1.00	1.05	1.06	0.85	1.06	0.99	1.05	1.06
0.10	1.11	0.95	1.12	0.99	1.11	1.12	0.85	1.13	0.98	1.11	1.14
0.15	1.18	0.95	1.19	0.99	1.18	1.19	0.85	1.21	0.97	1.18	1.22
0.20	1.25	0.95	1.27	0.99	1.25	1.27	0.85	1.31	0.96	1.26	1.32
0.25	1.33	0.95	1.36	0.98	1.34	1.37	0.85	1.42	0.95	1.36	1.44
0.30	1.43	0.95	1.46	0.98	1.44	1.47	0.85	1.55	0.94	1.47	1.57
0.35	1.54	0.95	1.58	0.97	1.56	1.60	0.85	1.70	0.92	1.61	1.74
0.40	1.67	0.95	1.73	0.97	1.70	1.75	0.85	1.89	0.91	1.78	1.95
0.45	1.82	0.95	1.90	0.97	1.87	1.93	0.85	2.13	0.90	1.99	2.21
0.50	2.00	0.95	2.11	0.96	2.08	2.16	0.85	2.43	0.89	2.28	2.55
0.60	2.50	0.95	2.71	0.96	2.68	2.80	0.85	3.40	0.87	3.22	3.70
0.70	3.33	0.95	3.80	0.95	3.80	4.01	0.85	5.67	0.85	5.70	6.72
0.80	5.00	0.95	6.33	0.94	6.62	7.02	0.85	17.00	0.83	30.2	36.6

P-δ effect and its role in the construction of the second-order stiffness.

Internal Second-Order Stiffness

Combining Equations 1 and 2 gives the following stiffness relationship:

$$\frac{H}{\Delta_2} = R_M \frac{H}{\Delta_1} - \frac{P_{story}}{L} \tag{22}$$

The three major terms in this equation may be considered to represent important system properties:

$\frac{H}{\Delta_2}$ = second-order stiffness, kip/in. (N/mm)

$\frac{H}{\Delta_1}$ = first-order stiffness, kip/in. (N/mm)

$\frac{P_{story}}{L}$ = P-Δ stiffness reduction, kip/in. (N/mm)

This P-Δ stiffness reduction is given the symbol $K_{P\Delta}$:

$$K_{P\Delta} = \frac{P_{story}}{L} \tag{23}$$

Equation 22 also includes the "internal second-order stiffness," the reduced lateral stiffness due to the P-δ effect of axial force on moment-frame columns (P_{mf}):

$R_M \frac{H}{\Delta_1}$ = internal second-order stiffness, kip/in. (N/mm)

This internal second-order stiffness corresponds to the internal force B_2H and the displacement Δ_2, as can be seen by combining Equation 8 with the definition above:

$$R_M \frac{H}{\Delta_1} = \frac{B_2 H}{\Delta_2} \tag{24}$$

Stiffness Equations Using P-Δ and P-δ Stiffness Reductions

The presence of the R_M term in Equation 22 appears to complicate the relationships between the first-order, second-order, and P-Δ stiffnesses. However, Equation 17 can be used to simplify the relationship. Applying the formulation of R_M from Equation 17 to the internal second-order stiffness results in:

$$R_M \frac{H}{\Delta_1} = \frac{H}{\Delta_1} - \frac{C_L P_{mf}}{L} \tag{25}$$

This last term is the stiffness-reduction effect of P-δ on the system, not including the P-Δ effect. It is termed the "P-δ stiffness reduction" and is given the symbol $K_{P\delta}$:

$$K_{P\delta} = \frac{C_L P_{mf}}{L} \tag{26}$$

Thus, the effect of P-δ on the system is not a multiplier on the first-order stiffness (as could be inferred from Equation 22). Instead, it can be described as a subtraction. The stiffness-sensitive formulation of R_M in Equation 17 simply reflects the fact that the importance of the reduction depends on its magnitude relative to the first-order stiffness.

The force and displacement amplifiers in Equations 10 and 15 can be presented in terms of first-order stiffness and the P-Δ, and P-δ stiffness reductions:

$$B_2 = \frac{H/\Delta_1 - K_{P\delta}}{H/\Delta_1 - (K_{P\Delta} + K_{P\delta})} \tag{27}$$

$$D_{AF} = \frac{1}{1 - \dfrac{K_{P\Delta} + K_{P\delta}}{H/\Delta_1}} \tag{28}$$

Combining Equations 22, 23, and 25 gives the external second-order stiffness:

$$\frac{H}{\Delta_2} = \frac{H}{\Delta_1} - (K_{P\Delta} + K_{P\delta}) \tag{29}$$

Equation 29 can be understood as signifying that the external stiffness in the presence of gravity loads is less than first-order stiffness due to the P-Δ and the P-δ stiffness reductions. Equation 29 can also be presented as:

$$\frac{H}{\Delta_1} = \frac{H}{\Delta_2} + K_{P\Delta} + K_{P\delta} \tag{30}$$

In this format, the equation signifies that the required first-order stiffness is the required external stiffness plus the P-Δ and the P-δ stiffness (applied as an addition, rather than as a reduction). This latter form has two important corollaries. First, if an engineer is designing a building to meet a drift limit, the magnitude of the gravity loads and their effect on the system stiffness requires a stiffer lateral system. Second, the required first-order stiffness may be determined in advance of design and analysis based on the drift limit, the geometry, and the external loading. Such a process is illustrated in Sabelli et al. (in review). This simple relationship also facilitates the use of hand methods to validate the results of computer analysis.

Note that the P-δ the stiffness reduction ($K_{P\delta}$) not only adds to the first-order stiffness required to meet a (second-order) drift limit in Equation 30. It also reduces the external stiffness (Equation 29) and thus increases the second-order drift Δ_2, indirectly contributing to the additional strength required by $K_{P\Delta}\Delta_2$ to resist the external force H.

As formulated in Equations 29 and 30, the first-order, second-order, P-Δ, and P-δ stiffness have a simple arithmetic relationship, which is diagrammed in Figure 2. The stiffnesses that correspond to the forces at the second-order drift Δ_2 are those in Equation 22: the external stiffness, which corresponds to the applied load H; the internal second-order stiffness, which corresponds to the total load effect B_2H; and the P-Δ stiffness reduction, which corresponds to the difference between the two. The first-order stiffness by definition corresponds to the first-order displacement Δ_1 resulting from the lateral load H. The other stiffnesses (the P-δ and total second-order stiffness reductions) are relevant in constructing the external stiffness but do not correspond directly to the load effect on the system.

As can be seen in Figure 2, the drift amplification from Δ_1 to Δ_2 does not produce a total load effect on the first-order stiffness line. Instead, the total load effect is somewhat less, falling on the internal second-order stiffness line. Thus, the drift amplifier is greater than the force amplifier when $K_{P\delta} > 0$ (and thus $R_M < 1.0$), consistent with Equation 6.

SUMMARY AND CONCLUSIONS

This paper presents the background for the AISC *Specification* factor R_M based on original work by LeMessurier (1977). It also presents a refined expression for R_M in Equation 17, which can be used to reduce conservatism resulting from simplifications in the AISC *Specification* (2016b) formulation of this quantity. The paper also presents the corresponding equation for B_2 in Equation 10; this is more accurate than the use of the *Specification* equation for R_M. A corresponding displacement amplifier (D_{AF}, presented in Equation 15) can be used to estimate second-order drift Δ_2 from the first-order drifts Δ_1. The refined formulation of R_M is used to present a clear relationship between the first-order, second-order, P-Δ, and P-δ stiffnesses.

The methods presented herein may be used by practicing engineers to reduce conservatism in design and to critically evaluate the results of computerized second-order analysis. The relationships developed may be beneficial to those learning the amplifier method and second-order effects more generally.

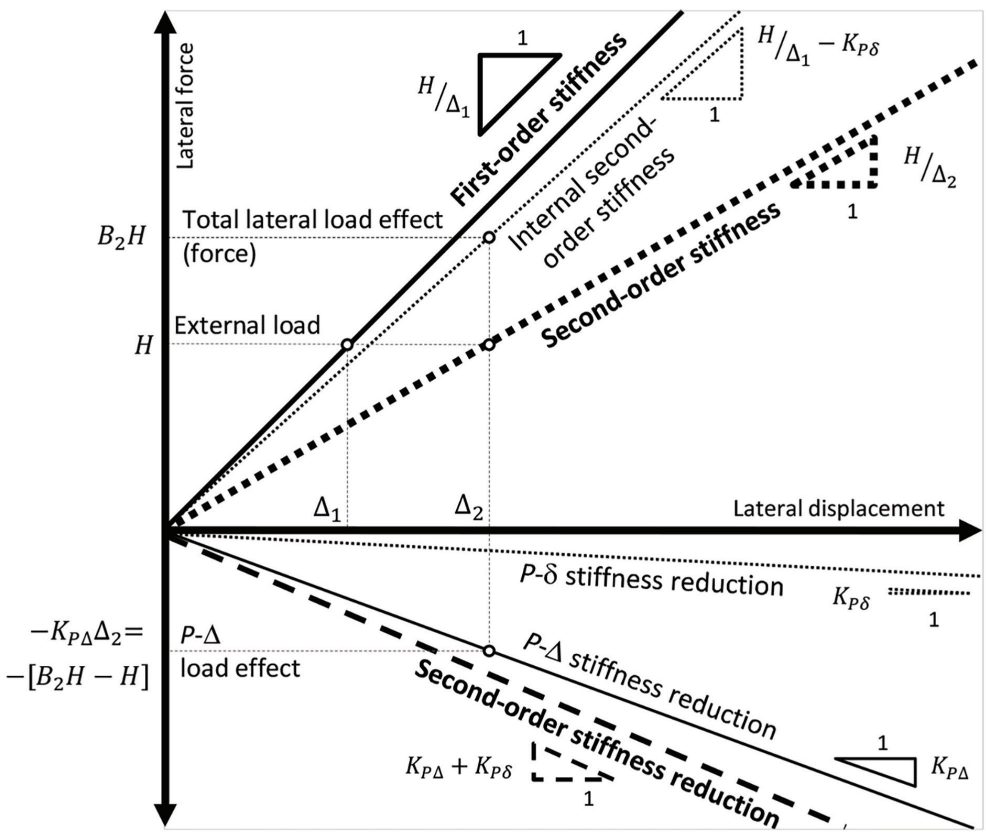

Fig. 2. First-order and second-order stiffness diagram.

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