Some Non-Conventional Cases of Column Design

SURESH T. DALAL

MOST PRACTICAL structural engineering papers deal with commonly encountered design problems—those likely to be found in conventional building or bridge structures. When engineers encounter uncommon problems, published guides are often lacking in numerical data and a great deal of time can be spent developing analytical solutions.

This paper deals with several non-conventional cases of elastic buckling of columns likely to be encountered by structural designers:

1. Symmetrically stepped columns with end axial loads
2. Unsymmetrically stepped columns with end axial loads
3. Prismatic columns with distributed axial loads
4. Prismatic columns with intermediate axial loads
5. Prismatic columns with end and intermediate axial loads
6. Unsymmetrically stepped columns with end and intermediate axial loads

SYMMETRICALLY STEPPED COLUMNS WITH END LOADS

Critical elastic buckling loads for symmetrically stepped columns with hinged ends are presented in Fig. 1 and Table 1.

Let L = column length
A = Length of the center segment
EI_2 = Flexural rigidity of the center segment in the plane of buckling
EI_1 = Flexural rigidity of the end segments in the plane of buckling
α = Ratio A/L
β = Ratio I_2/I_1
P_{cr} = Critical value of the end load

Suresh T. Dalal is a structural design engineer in Oakland, Calif.

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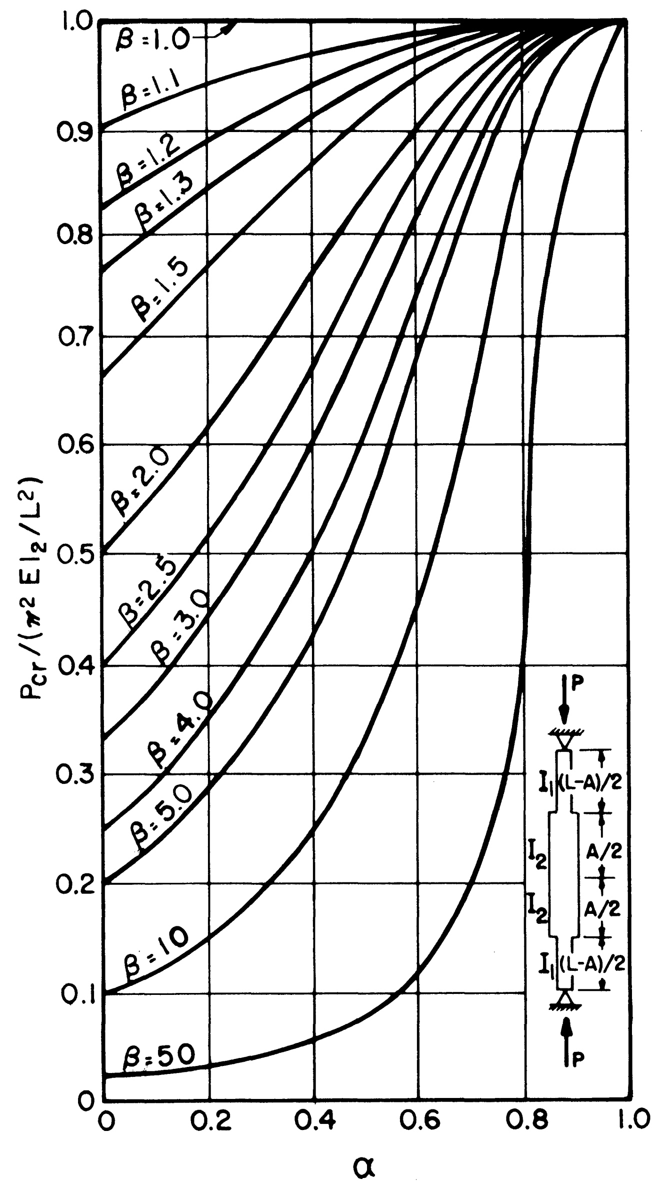

Fig. 1. Symmetrically stepped columns

Table 1. Symmetrically Stepped Columns with Hinged Ends

I_2/I_1	A/L	L_{eff}/L	P_{cr}/P_e	I_2/I_1	A/L	L_{eff}/L	P_{cr}/P_e
1.00	0	1.000000	1.000000	2.50	0	1.581139	0.400000
	0.1	1.000000	1.000000		0.1	1.486412	0.452607
	0.2	1.000000	1.000000		0.2	1.392700	0.515567
	0.3	1.000000	1.000000		0.3	1.301612	0.590251
	0.4	1.000000	1.000000		0.4	1.215756	0.676561
	0.5	1.000000	1.000000		0.5	1.139128	0.770647
	0.6	1.000000	1.000000		0.6	1.076937	0.862223
	0.7	1.000000	1.000000		0.7	1.033731	0.935804
	0.8	1.000000	1.000000		0.8	1.010048	0.980202
	0.9	1.000000	1.000000		0.9	1.001243	0.997519
	1.0	1.000000	1.000000		1.0	1.000000	1.000000
1.10	0	1.048809	0.909091	3.00	0	1.732051	0.333333
	0.1	1.039340	0.925731		0.1	1.616702	0.382596
	0.2	1.030270	0.942103		0.2	1.502243	0.443118
	0.3	1.021991	0.957428		0.3	1.390166	0.517448
	0.4	1.014852	0.970945		0.4	1.283080	0.607425
	0.5	1.009110	0.982027		0.5	1.185418	0.711635
	0.6	1.004884	0.990302		0.6	1.103889	0.820633
	0.7	1.002133	0.995748		0.7	1.045766	0.914390
	0.8	1.000647	0.998708		0.8	1.013566	0.973410
	0.9	1.000082	0.999836		0.9	1.001664	0.996681
	1.0	1.000000	1.000000		1.0	1.000000	1.000000
1.20	0	1.095445	0.833333	4.00	0	2.000000	0.250000
	0.1	1.077295	0.861649		0.1	1.850090	0.292156
	0.2	1.059814	0.890310		0.2	1.700860	0.345671
	0.3	1.043709	0.917997		0.3	1.553531	0.414343
	0.4	1.029671	0.943198		0.4	1.410433	0.502684
	0.5	1.018265	0.964446		0.5	1.276075	0.614112
	0.6	1.009810	0.980665		0.6	1.158802	0.744700
	0.7	1.004282	0.991490		0.7	1.070901	0.871970
	0.8	1.001297	0.997412		0.8	1.020860	0.959550
	0.9	1.000164	0.999672		0.9	1.002516	0.994986
	1.0	1.000000	1.000000		1.0	1.000000	1.000000
1.30	0	1.140175	0.769231	5.00	0	2.236068	0.200000
	0.1	1.113998	0.805808		0.1	2.057252	0.236279
	0.2	1.088670	0.843738		0.2	1.878969	0.283244
	0.3	1.065150	0.881410		0.3	1.702214	0.345121
	0.4	1.044447	0.916700		0.4	1.528978	0.427757
	0.5	1.027461	0.947259		0.5	1.363414	0.537953
	0.6	1.014775	0.971092		0.6	1.214230	0.678262
	0.7	1.006449	0.987226		0.7	1.097292	0.830531
	0.8	1.001950	0.996111		0.8	1.028504	0.945341
	0.9	1.000246	0.999508		0.9	1.003383	0.993268
	1.0	1.000000	1.000000		1.0	1.000000	1.000000
1.50	0	1.224745	0.666667	10.00	0	3.162278	0.100000
	0.1	1.184080	0.713244		0.1	2.877701	0.120756
	0.2	1.144462	0.763479		0.2	2.593348	0.148689
	0.3	1.107196	0.815739		0.3	2.309664	0.187457
	0.4	1.073826	0.867225		0.4	2.027624	0.243234
	0.5	1.045951	0.914065		0.5	1.749512	0.326713
	0.6	1.024819	0.952151		0.6	1.481327	0.455720
	0.7	1.010832	0.978682		0.7	1.240639	0.649694
	0.8	1.003266	0.993499		0.8	1.072023	0.870145
	0.9	1.000411	0.999179		0.9	1.007938	0.984311
	1.0	1.000000	1.000000		1.0	1.000000	1.000000
2.00	0	1.414214	0.500000	50.00	0	7.071068	0.020000
	0.1	1.343664	0.553884		0.1	6.378106	0.024582
	0.2	1.274234	0.615888		0.2	5.685167	0.030940
	0.3	1.207546	0.685793		0.3	4.992298	0.040124
	0.4	1.145944	0.761506		0.4	4.299607	0.054093
	0.5	1.092509	0.837818		0.5	3.607360	0.076846
	0.6	1.050523	0.906127		0.6	2.916297	0.117581
	0.7	1.022081	0.957259		0.7	2.229005	0.201269
	0.8	1.006615	0.986899		0.8	1.559425	0.411217
	0.9	1.000825	0.998352		0.9	1.061945	0.886739
	1.0	1.000000	1.000000		1.0	1.000000	1.000000

Note: P_e and L_{eff} are referred to I_2.

The buckling equation for this column is given by:

$$\tan \frac{(K_1L)(1-\alpha)}{2} \cdot \tan \frac{(K_1L)(\alpha)}{2\sqrt{\beta}} = \sqrt{\beta} \quad (1)$$

where

$$K_1^2 = \frac{P}{EI_1} \quad (2)$$

Buckling loads for various values of β (≥ 1) and α were calculated by solving Equation (1) with the help of a digital computer using the method of linear interpolation ("Regula Falsi").[1] Values were then normalized by dividing by the corresponding values of the Euler buckling load P_e for a prismatic column of length L and moment of inertia I_2. Also, effective length factors were calculated based on an equivalent prismatic column with moment of inertia I_2.

Thus:

$$P_e = \frac{\pi^2 EI_2}{L^2} \quad (3)$$

$$P_{cr} = \frac{\pi^2 EI_2}{L_{eff}^2} \quad (4)$$

or

$$\frac{L_{eff}}{L} = \frac{\pi\sqrt{\beta}}{K_1L} \quad (5)$$

Two special cases can be recognized:

(a) When the column is prismatic with moment of inertia I_1:

$$\alpha = 0$$

$$\frac{L_{eff}}{L} = \sqrt{\beta}$$

$$\frac{P_{cr}}{P_e} = \frac{1}{\beta}$$

(b) When the column is prismatic with moment of inertia I_2:

$$\alpha = 1 \quad \text{and/or} \quad \beta = 1$$

$$\frac{L_{eff}}{L} = 1.0$$

$$\frac{P_{cr}}{P_e} = 1.0$$

The values of P_{cr} given here agree with those given in a similar table in Reference 2, the scope of which table has been extended here. It may be noted that the above data can also be used for an unsymmetrical stepped cantilever column (with fixed support at the end of the segment having moment of inertia I_2), if Equations (3) and (4) are modified as follows:

$$P_e = \frac{\pi^2 EI_2}{4L^2} \quad (3a)$$

$$P_{cr} = \frac{\pi^2 EI_2}{4L_{eff}^2} \quad (4a)$$

The ratio L_{eff}/L is unaffected, and Equation (5) still holds. Similarly values of P_{cr}/P_e are also unaffected.

Example:

Given: A stepped column (Fig. 1) with hinged ends: $E = 30{,}000$ kips/in.^{2}, $L = 200$ in., $A = 120$ in., $I_1 = 50$ in.^{4}, $I_2 = 100$ in.^{4}.

Find the critical buckling load for this column.

Solution:

$$\alpha = \frac{A}{L} = \frac{120}{200} = 0.6$$

$$\beta = \frac{I_2}{I_1} = \frac{100}{50} = 2.0$$

The Euler buckling load for a prismatic column with $I_2 = 100$ in.^{4} is:

$$P_e = \frac{\pi^2 EI_2}{L^2} = \frac{9.8696 \times 30{,}000 \times 100}{(200)^2} = 740.2 \text{ kips}$$

From Table 1 (or Fig. 1):

$$\frac{P_{cr}}{P_e} = 0.9061$$

$$P_{cr} = 0.9061 \times 740.2 = 670.7 \text{ kips}$$

Alternatively:

$$\frac{L_{eff}}{L} = 1.0505 \text{ (from Table 1)}$$

$$L_{eff} = 200 \times 1.0505 = 210.1 \text{ in.}$$

$$P_{cr} = \frac{\pi^2 EI_2}{L_{eff}^2} = \frac{9.8696 \times 30{,}000 \times 100}{(210.1)^2} = 670.7 \text{ kips}$$

UNSYMMETRICALLY STEPPED HINGED COLUMNS WITH END LOADS

Elastic buckling loads for this case are given in Fig. 2 and Table 2.

Nomenclature for this case is the same as for the symmetrical case, except that:

A = Length of lower segment
EI_2 = Flexural rigidity of the lower segment in the plane of buckling
EI_1 = Flexural rigidity of the upper segment in the plane of bending.

The buckling equation for this column is given by:[2]

$$\sqrt{\beta}\tan\left[\sqrt{\frac{1}{\beta}}(K_1L)(\alpha)\right] + \tan[(K_1L)(1-\alpha)] = 0 \quad (6)$$

where $K_1^2 = P_{cr}/EI_1$.

Equation (6) was solved for values of $0 \leq \alpha \leq 1.0$ and $\beta \geq 1.0$, using a digital computer.[1] Critical loads, P_{cr}, calculated therefrom, were normalized by dividing by P_e, where

$$P_e = \pi^2 EI_2/L^2$$

is the value of Euler buckling load for a prismatic column of moment of inertia I_2 and length L.

Effective length factors for an equivalent prismatic column with moment of inertia I_2 were also calculated:

$$\frac{L_{eff}}{L} = \frac{\pi\sqrt{\beta}}{K_1L} \quad (5)$$

Two special cases involving known results can be deduced directly from Equation (6):

(a) When the column has a uniform moment of inertia I_2:

$$\alpha = 1.0 \quad \text{and/or} \quad \beta = 1.0$$

$$L_{eff}/L = 1.0 \quad \text{and} \quad P_{cr}/P_e = 1.0$$

(b) When the column has uniform moment of inertia I_1:

$$\alpha = 0$$

$$L_{eff}/L = \sqrt{\beta} \quad \text{and} \quad P_{cr}/P_e = 1/\beta$$

Example:

Given: A hinged stepped column with $E = 30{,}000$ kip/in.^{2}, $L = 200$ in., $A = 100$ in., $I_1 = 60$ in.^{4}, $I_2 = 90$ in.^{4}.

Find the critical buckling load for this column.

Solution:

$$\alpha = \frac{A}{L} = \frac{100}{200} = 0.5$$

$$\beta = \frac{I_2}{I_1} = \frac{90}{60} = 1.5$$

The Euler buckling load for a prismatic column with $I_2 = 90$ in.^{4} is

$$P_e = \frac{\pi^2 EI_2}{L^2} = \frac{9.8696 \times 30{,}000 \times 90}{(200)^2} = 666.2 \text{ kips}$$

From Table 2 (or Fig. 2):

$$P_{cr}/P_e = 0.792$$

$$P_{cr} = 0.792 \times 666.2 = 527.6 \text{ kips}$$

Alternatively,

$$L_{eff}/L = 1.1235 \text{ (from Table 2)}$$

$$L_{eff} = 200 \times 1.1235 = 224.7 \text{ in.}$$

$$P_{cr} = \frac{\pi^2 EI_2}{L_{eff}^2} = \frac{9.8696 \times 30{,}000 \times 90}{(224.7)^2} = 527.8 \text{ kips}$$

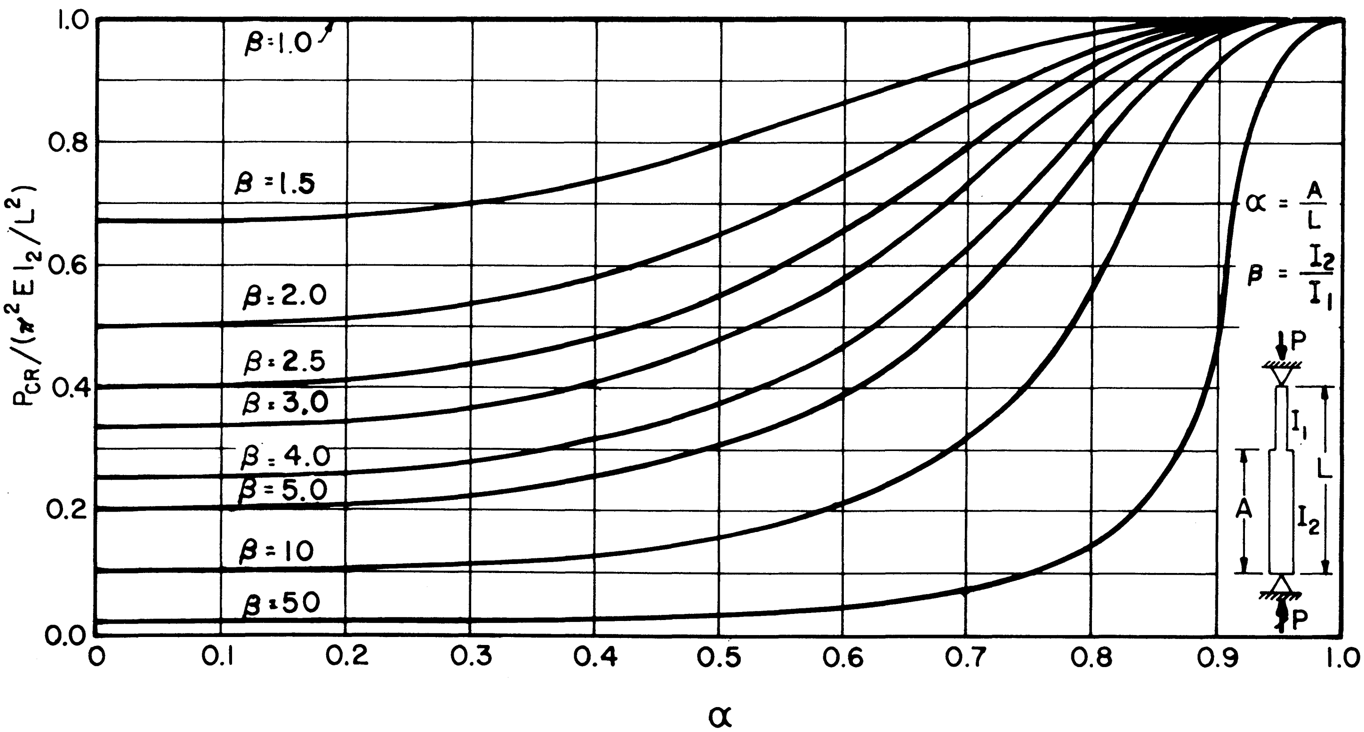

Fig. 2. Unsymmetrically stepped columns

Table 2. Unsymmetrically Stepped Columns with Hinged Ends

I_2/I_1	A/L	L_{eff}/L	P_{cr}/P_e
1.00	0	1.000000	1.000000
	0.1	1.000000	1.000000
	0.2	1.000000	1.000000
	0.3	1.000000	1.000000
	0.4	1.000000	1.000000
	0.5	1.000000	1.000000
	0.6	1.000000	1.000000
	0.7	1.000000	1.000000
	0.8	1.000000	1.000000
	0.9	1.000000	1.000000
	1.0	1.000000	1.000000
1.50	0	1.224745	0.666667
	0.1	1.223442	0.668088
	0.2	1.215107	0.677284
	0.3	1.195615	0.699548
	0.4	1.164286	0.737701
	0.5	1.123540	0.792178
	0.6	1.078963	0.858987
	0.7	1.039173	0.926028
	0.8	1.012702	0.975072
	0.9	1.001639	0.996731
	1.0	1.000000	1.000000
2.00	0	1.414214	0.500000
	0.1	1.411968	0.501591
	0.2	1.397767	0.511835
	0.3	1.364848	0.536824
	0.4	1.311756	0.581158
	0.5	1.241077	0.649236
	0.6	1.159806	0.743411
	0.7	1.081744	0.854577
	0.8	1.026507	0.949022
	0.9	1.003330	0.993373
	1.0	1.000000	1.000000
2.50	0	1.581139	0.400000
	0.1	1.578136	0.401524
	0.2	1.559271	0.411298
	0.3	1.515797	0.435229
	0.4	1.445675	0.478474
	0.5	1.351329	0.547618
	0.6	1.239755	0.650621
	0.7	1.126682	0.787767
	0.8	1.041422	0.922033
	0.9	1.005076	0.989925
	1.0	1.000000	1.000000
3.00	0	1.732051	0.333333
	0.1	1.728403	0.334742
	0.2	1.705593	0.343755
	0.3	1.653246	0.365868
	0.4	1.568876	0.406278
	0.5	1.454745	0.472527
	0.6	1.317503	0.576098
	0.7	1.173019	0.726758
	0.8	1.057425	0.894335
	0.9	1.006879	0.986383
	1.0	1.000000	1.000000
4.00	0	2.000000	0.250000
	0.1	1.995274	0.251186
	0.2	1.965886	0.258752
	0.3	1.898815	0.277354
	0.4	1.790913	0.311782
	0.5	1.644268	0.369875
	0.6	1.464850	0.466030
	0.7	1.266970	0.622970
	0.8	1.092479	0.837865
	0.9	1.010663	0.979011
	1.0	1.000000	1.000000
5.00	0	2.236068	0.200000
	0.1	2.230441	0.201010
	0.2	2.195567	0.207447
	0.3	2.116250	0.223288
	0.4	1.988846	0.252812
	0.5	1.815418	0.303422
	0.6	1.601526	0.389881
	0.7	1.359682	0.540910
	0.8	1.131008	0.781751
	0.9	1.014699	0.971238
	1.0	1.000000	1.000000
10.00	0	3.162278	0.100000
	0.1	3.153353	0.100567
	0.2	3.098418	0.104165
	0.3	2.974363	0.113035
	0.4	2.775939	0.129772
	0.5	2.505679	0.159276
	0.6	2.169106	0.212539
	0.7	1.774785	0.317474
	0.8	1.347052	0.551101
	0.9	1.039274	0.925848
	1.0	1.000000	1.000000
50.00	0	7.071068	0.020000
	0.05	7.068257	0.020016
	0.10	7.049390	0.020123
	0.15	7.001681	0.020398
	0.20	6.916701	0.020903
	0.25	6.789761	0.021692
	0.30	6.618820	0.022826
	0.35	6.403520	0.024387
	0.40	6.144443	0.026487
	0.45	5.842702	0.029294
	0.50	5.499653	0.033062
	0.55	5.116769	0.038195
	0.60	4.695528	0.045356
	0.65	4.237391	0.055693
	0.70	3.743804	0.071347
	0.75	3.216261	0.096671
	0.80	2.656558	0.141697
	0.85	2.067920	0.233848
	0.90	1.463876	0.466650
	0.95	1.034019	0.935282
	1.00	1.000000	1.000000

Note: L_{eff} and P_{cr} are referred to I_2.

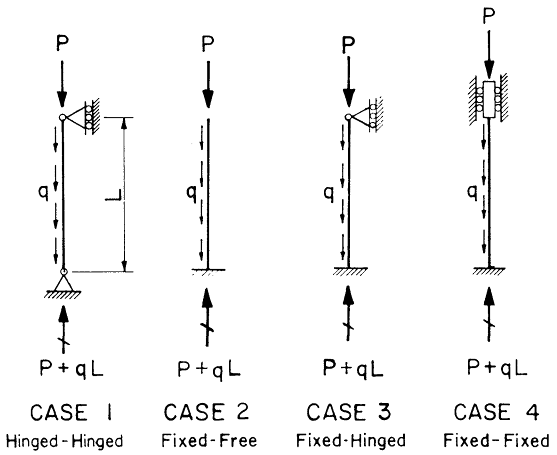

Figure 3

PRISMATIC COLUMNS WITH DISTRIBUTED AXIAL LOADS

Critical buckling loads for prismatic columns with distributed axial loads acting in combination with an end load are presented here in both graphical and numerical form. The results are "exact" within the limitations of the ordinary elastic buckling theory.

The four different end conditions considered are shown in Fig. 3. In each case, the loading consists of a uniformly distributed axial force q per unit length of the column and a concentrated force P at the upper end of the column. Both q and P may be compressive (positive), or either one may be a tensile (negative) load while the other is compressive. However, in all cases it is the critical combination of the two loads that represents the conditions for elastic buckling. The loads are assumed to remain vertical during buckling.

The differential equation of the buckled shape of a prismatic column with generalized end conditions (so that axial, shear and moment reactions may exist at both ends) is given by:

$$\frac{d^4y}{dx^4} + \left(\frac{\alpha + \beta}{L^2}\right)\frac{d^2y}{dx^2} - \left(\frac{\alpha}{L^2}\right)\left(\frac{x}{L}\right)\frac{d^2y}{dx^2} - \left(\frac{\alpha}{L^3}\right)\frac{dy}{dx} = 0 \quad (7)$$

where $\alpha = qL^3/EI$ and $\beta = PL^2/EI$.

Equation (7) is a fourth-order homogeneous equation with a variable coefficient, and the solution involves integrals which cannot be evaluated in closed form. The "series solution method of Frobenius" was used to obtain the solution in the form:

$$y = a_0 + a_1S_1 + a_2S_2 + a_3S_3 \quad (8)$$

in which the a's are constants, and S_1, S_2, S_3 are the infinite series given by

$$S_1 = \frac{x}{L}\left[1 + \frac{\alpha}{4!}\left(\frac{x}{L}\right)^3 - \frac{\alpha(\alpha+\beta)}{6!}\left(\frac{x}{L}\right)^5 + \ldots\right] \quad (9)$$

$$S_2 = \left(\frac{x}{L}\right)^2\left[1 - \frac{2(\alpha+\beta)}{4!}\left(\frac{x}{L}\right)^2 + \frac{4\alpha}{5!}\left(\frac{x}{L}\right)^3 + \ldots\right] \quad (10)$$

$$S_3 = \left(\frac{x}{L}\right)^3\left[1 - \frac{3!(\alpha+\beta)}{5!}\left(\frac{x}{L}\right)^2 + \frac{3(3!)\alpha}{6!}\left(\frac{x}{L}\right)^3 + \ldots\right] \quad (11)$$

When the appropriate boundary conditions are substituted in Equation (8), four equations are obtained with a_0, a_1, a_2, and a_3 as unknowns. Elimination of these constants leads to the buckling equation from which the critical buckling loads can be found. Because the buckling equation involves infinite series, a digital computer was used to compute the desired roots and hence the buckling loads. The method of linear interpolation ("Regula Falsi") was used for this purpose.

The boundary conditions for the various end-conditions shown in Fig. 3 can be expressed as follows:

(a) Pinned support: $y = 0$ and $\frac{d^2y}{dx^2} = 0$

(b) Fixed support: $y = 0$ and $\frac{dy}{dx} = 0$

(c) Free end: $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} + \left(\frac{\beta}{L^2}\right)\frac{dy}{dx} = 0$

The results of using these boundary conditions for the four cases shown in Fig. 3 are given in Fig. 4 and Table 3. Two special cases can be recognized:

1. If $q = 0$, the column buckles at a load $P_{cr} = P_e$, where P_e is the Euler buckling load
2. If $P = 0$, buckling is due to distributed axial force only, its value being denoted by $(qL)_{cr}$.

If qL is larger than this critical value, the end force P_{cr} is negative (tensile). If qL is negative, then P_{cr} exceeds the Euler load, P_e.

With the aid of the data presented in Fig. 4 and Table 3, the critical combinations of concentrated end load P and distributed axial load q can be found in each particular case of end conditions. The only data of this type previously available were limited to Cases 1 and 2, and to positive values of q. For the special case of $P = 0$, the buckling loads determined above agree with previous results.

In the range $-4 \leq qL/P_e \leq 10$, the interaction effect of the uniform load qL on the magnitude of P_{cr} can be represented by the following empirical relations, in which $m = P_{cr}/P_e$ and $n = qL/P_e$:

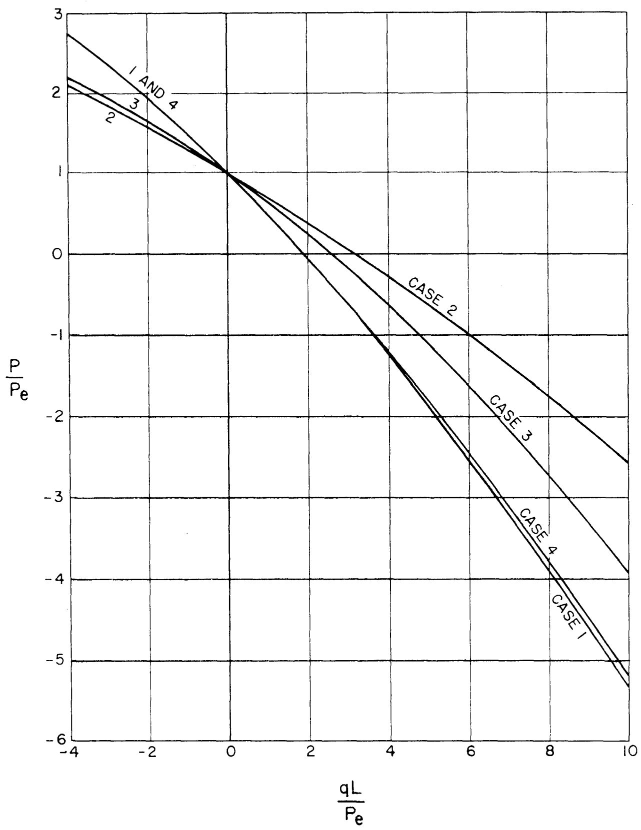

Fig. 4. Prismatic columns with distributed axial loads

Table 3. Columns with Distributed and End Axial Loads

$\frac{qL}{P_e}$	P_{cr}/P_e			
	Case 1	Case 2	Case 3	Case 4
−4.00	2.7426	2.1134	2.2040	2.7658
−3.00	2.3520	1.8484	1.9319	2.3649
−2.00	1.9331	1.5749	1.6426	1.9387
−1.50	1.7121	1.4348	1.4905	1.7153
−1.00	1.4831	1.2923	1.3330	1.4845
−0.50	1.2458	1.1474	1.1697	1.2461
0.00	1.0000	1.0000	1.0000	1.0000
0.25	0.8739	0.9253	0.9127	0.8740
0.50	0.7458	0.8500	0.8236	0.7461
0.75	0.6155	0.7740	0.7327	0.6163
1.00	0.4831	0.6974	0.6400	0.4845
1.50	0.2121	0.5420	0.4488	0.2153
2.00	−0.0669	0.3838	0.2497	−0.0613
2.50	−0.3537	0.2227	0.0425	−0.3448
3.00	−0.6480	0.0586	−0.1730	−0.6351
3.50	−0.9493	−0.1085	−0.3968	−0.9317
4.00	−1.2574	−0.2788	−0.6289	−1.2342
5.00	−1.8917	−0.6289	−1.1172	−1.8554
6.00	−2.5479	−0.9922	−1.6354	−2.4959
8.00	−3.9143	−1.7600	−2.7479	−3.8246
10.00	−5.3373	−2.5843	−3.9403	−5.2044
$\frac{(qL)cr}{P_e}$ for $P = 0$	1.8814	3.1764	2.6002	1.8904

Case 1—Hinged-Hinged:

$$m = 1.0000 - 0.5000n - 0.01692n^2 + 0.000051n^4 - 0.00000015n^6$$

(max. error of −0.10% at $n = 10$)

Case 2—Fixed-Free:

$$m = 1.0000 - 0.2973n - 0.00520n^2 - 0.00010n^3$$

(max. error of +0.34% at $n = 10$)

Case 3—Fixed-Hinged:

$$m = 1.0000 - 0.3465n - 0.01334n^2 - 0.00056n^3 + 0.0000136n^4 + 0.00000285n^5$$

(max. error of +0.56% at $n = 8$)

Case 4—Fixed-Fixed:

$$m = 1.0000 - 0.5001n - 0.01550n^2 + 0.000073n^3 + 0.000030n^4$$

(max. error of −0.51% at $n = 10$)

Examples:

Given: A Fixed-Hinged column (Case 3, Fig. 3): E = 30,000 kips/in.², I = 100 in.⁴, L = 150 in. The column carries a uniformly distributed compressive load q of 9 kips/in. along its length. Find the maximum end load P it can carry before buckling.

Solution:

$$P_e = \frac{20.1907 \times 30{,}000 \times 100}{(150)^2} = 2{,}692 \text{ kips}$$

Total distributed load,

$$qL = 9 \times 150 = 1{,}350 \text{ kips}$$

$$n = qL/P_e = 1{,}350/2{,}692 = 0.501$$

From Table 3 (or Fig. 4 or the empirical relation):

$$m = P_{cr}/P_e = 0.8233$$

$$P_{cr} = 0.8233 \times 2{,}692 = 2{,}216 \text{ kips (compressive)}$$

Given: A cantilever column (Case 2, Fig. 3): E = 30,000 kips/in.², I = 200 in.⁴, L = 100 in. The column carries an end load P of 1,145 kips. Find the maximum uniformly distributed load q it can carry before buckling.

Solution:

$$P_e = \frac{\pi^2 EI}{4L^2} = \frac{9.8696 \times 30{,}000 \times 200}{(4 \times 100)^2} = 1{,}480 \text{ kips}$$

$$m = P/P_e = 1{,}145/1{,}480 = 0.774$$

From Table 3 (or Fig. 4):

$$n = qL/P_e = 0.75$$

$$qL = (0.75)(1{,}480) = 1{,}110 \text{ kips}$$

$$q = 11.1 \text{ kips/in. (compressive)}$$

PRISMATIC COLUMNS WITH INTERMEDIATE AXIAL LOADS

For the case of a hinged prismatic column with intermediate axial load and no end load at the top, the buckling equation[1,2] is given by:

$$\tan\left[(KL)\left(\frac{A}{L}\right)\right] = \frac{(KL)(1 - A/L)^2}{(A/L) - 2 + {}^1/_3(KL)^2(1 - A/L)^3} \qquad (12)$$

where L = length of column,

A = height of the point of application of the load, P, above the base,

$K^2 = P/EI$

EI = Flexural rigidity of the column in the plane of buckling

Equation (12) was solved with the help of a digital computer for various values of A/L ($0 \leq A/L \leq 1$). The critical loads, P_{cr}, calculated therefrom were then normalized by dividing them by the Euler buckling load, P_e, where:

$$P_e = \frac{\pi^2 EI}{L^2} \qquad (13)$$

The results are given in Table 4 and a plot of P_{cr}/P_e vs. A/L is shown in Fig. 5.

Two special conditions can be deduced directly from Equation (12):

(a) When $A = L$, the load is applied at the top end, and $P_{cr} = P_e$, the Euler buckling load

(b) When $A = 0$, the load is applied at the base, and $P_{cr} = \infty$, i.e., the column can carry an infinitely large load at the base before it starts buckling, as is obvious by inspection.

Table 4. Hinged Prismatic Columns with an Intermediate Axial Load *P* at Height *A* Above Base (No End Load at Top)

A/L	P_{cr}/P_e	A/L	P_{cr}/P_e
0	Infinity		
0.05	6.732247	0.55	1.874724
0.10	3.743081	0.60	1.831386
0.15	2.786414	0.65	1.758470
0.20	2.343013	0.70	1.662178
0.25	2.109262	0.75	1.552196
0.30	1.983632	0.80	1.436916
0.35	1.921366	0.85	1.321834
0.40	1.897396	0.90	1.209920
0.45	1.893182	0.95	1.102498
0.50	1.891248	1.00	1.000000

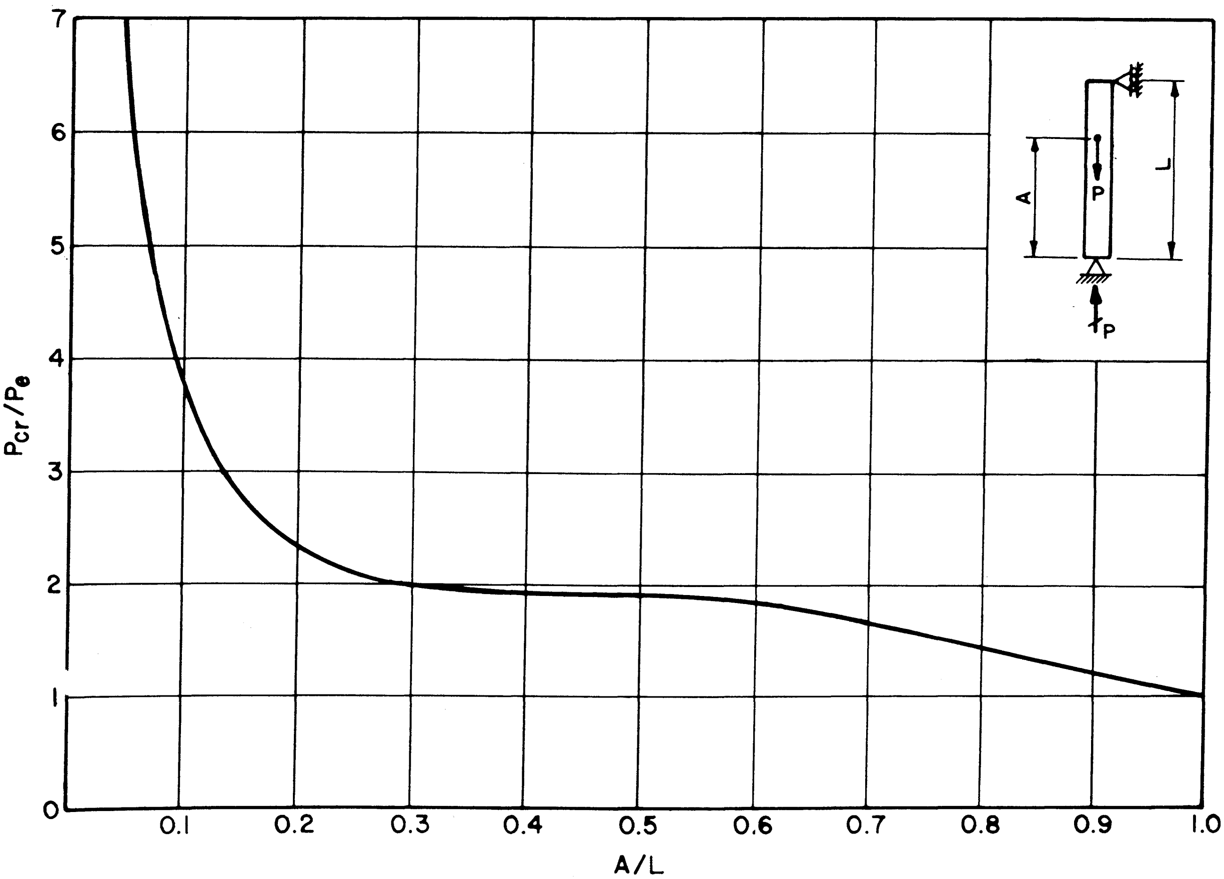

Fig. 5. Prismatic columns with intermediate axial loads

PRISMATIC COLUMNS WITH END AND INTERMEDIATE LOADS

For the case of a hinged prismatic column with two equal loads, one at the top and one at intermediate height, the condition of elastic buckling can be expressed by:

$$\frac{2}{\tan[(KL)(1-A)]} + \frac{\sqrt{2}}{\tan[(\sqrt{2}A)(KL)]} = \frac{1}{(KL)(2-A)} \qquad (14)$$

where the parameters are as defined previously.

This buckling equation was also solved with the help of a digital computer using the method of linear interpolation.[1] The critical values, P_{cr}, calculated therefrom were nondimensionalized by dividing them by (EI/L^2). The effective length factors, (L_{eff}/L), were then calculated such that the total load, $(2P)_{cr}$, is given by the expression:

$$(2P)_{cr} = \pi^2 EI/L^2 \qquad (15)$$

The results are given in Table 5. Figure 6 is a plot of $P_{cr}/(EI/L^2)$ vs. A/L.

Table 5. Hinged Prismatic Columns with Two Equal Loads *P*, One at Top, One at Height *A* Above Base

A/L	L_{eff}/L	$P_{cr}/(EI/L^2)$
0	0.707107	9.869604
0.05	0.743198	8.934302
0.10	0.777356	8.166393
0.15	0.806427	7.588215
0.20	0.829349	7.174564
0.25	0.846201	6.891646
0.30	0.857603	6.709601
0.35	0.864445	6.603812
0.40	0.867787	6.553057
0.45	0.868832	6.537283
0.50	0.868917	6.536020
0.55	0.869439	6.528170
0.60	0.871736	6.493817
0.65	0.876883	6.417805
0.70	0.885517	6.293258
0.75	0.897781	6.122508
0.80	0.913416	5.914703
0.85	0.931949	5.681792
0.90	0.952856	5.435190
0.95	0.975665	5.184038
1.00	1.000000	4.934802

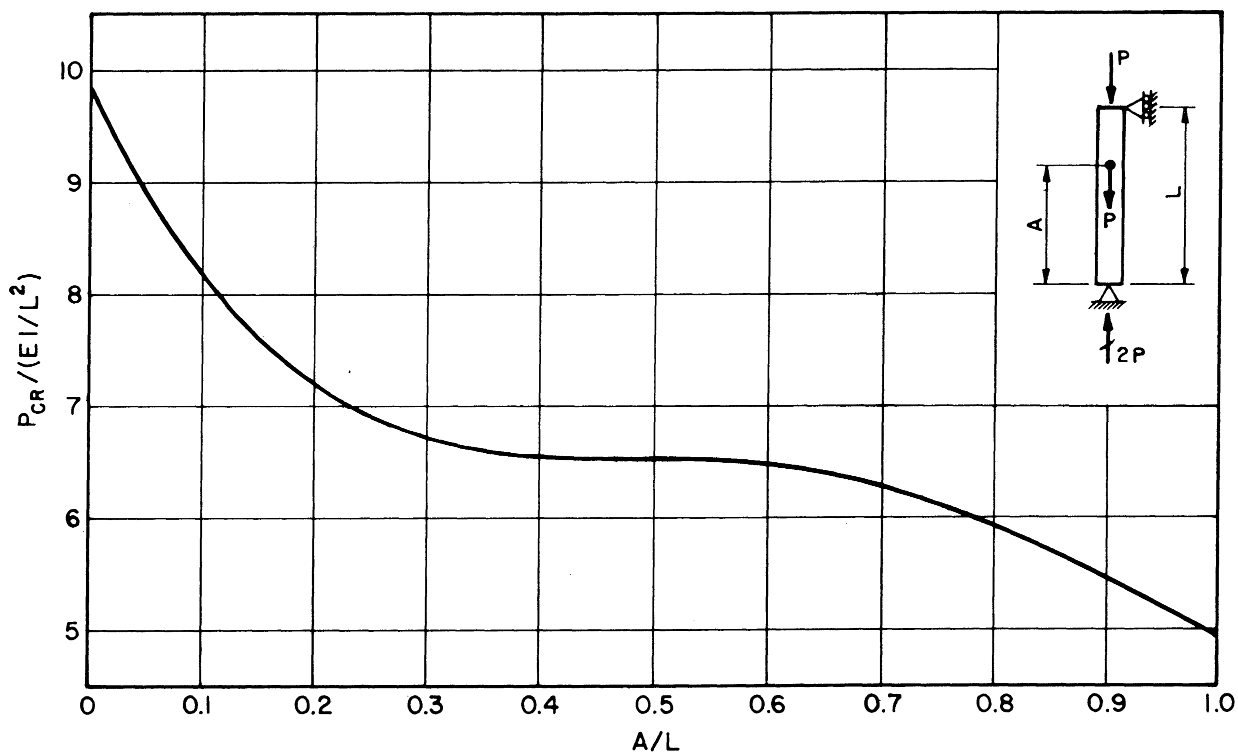

Fig. 6. Prismatic columns with end and intermediate axial loads

Again, two special conditions can be recognized:

(a) When $A/L = 0$, one load is at the top end and the other at the base, and

$$P_{cr} = \text{Euler buckling load}$$
$$= \pi^2 EI/L^2 = 9.868604\ (EI/L^2) \qquad (16)$$

$$KL = \pi$$

$$\frac{L_{eff}}{L} = \frac{\pi}{\sqrt{2}(KL)} = \frac{1}{\sqrt{2}} = 0.707107 \qquad (17)$$

(b) When $A/L = 1$, both loads are applied at the top end, and

$$(2P)_{cr} = \text{Euler buckling load} = \pi^2 EI/L^2$$
$$P_{cr} = \tfrac{1}{2}\ (\pi^2 EI/L^2) = 4.934802\ (EI/L^2) \qquad (18)$$

In this case,

$$KL = \pi/\sqrt{2}$$

so that

$$L_{eff}/L = \pi/(\sqrt{2}KL) = 1.0 \qquad (19)$$

Example:

Given: A hinged column with: E = 30,000 kips/in.², $I = 100$ in.⁴, $L = 150$ in.

The column carries two equal loads; one at the top and the other at 90 in. above the base. Find the critical value of these loads.

Solution:

$$A/L = 90/150 = 0.6$$

From Table 4 (or Fig. 5):

$$P_{cr}/(EI/L^2) = 6.494$$

$$P_{cr} = \frac{6.494 \times 30{,}000 \times 100}{(150)^2} = 865.9 \text{ kips}$$

Alternatively, from Table 4, for $A/L = 0.6$:

$$L_{eff}/L = 0.872$$

$$L_{eff} = 0.872 \times 150 = 130.8 \text{ in.}$$

$$(2P)_{cr} = \pi^2 EI/L_{eff}^2$$
$$= \frac{\pi^2 \times 30{,}000 \times 100}{(130.8)^2} = 1730.6 \text{ kips}$$

$$P_{cr} = 865.3 \text{ kips}$$

STEPPED COLUMNS WITH INTERMEDIATE LOADS

The case of a non-prismatic column with hinged ends, carrying two loads, P_1 at the top end and P_2 at mid-height, is not uncommon.

Let the cross-sectional moment of inertia of the upper half be I_1 and that of the lower half be I_2. Let $\alpha = P_2/P_1$ and $\beta = I_2/I_1$.

The buckling equation[2] for this column is given by:

$$\frac{2\alpha^2}{\alpha + 2}\sqrt{\frac{\beta}{1+\alpha}} = (K_1L)\left[\frac{\sqrt{\beta(1+\alpha)}}{\tan(K_1L/2)} + \frac{1}{\tan(\sqrt{1+\alpha/\beta}\cdot K_1L/2)}\right] \qquad (20)$$

where $K_1^2 = P_1/EI_1 \qquad (21)$

This equation was solved with the help of a digital computer for various values of α and β, and the critical values of the total load, $(P_1 + P_2)_{cr}$, were calculated. These were nondimensionalized by dividing by EI_2/L^2.

The critical values can also be represented as[2]:

$$(P_1 + P_2)_{cr} = \frac{\pi^2 EI_2}{L^2_{eff}} \text{ or } \frac{L_{eff}}{L} = \frac{\pi\sqrt{\beta}}{(K_1L)\sqrt{1+\alpha}} \qquad (22)$$

where L_{eff} is in terms of I_2.

Values of $(P_1 + P_2)_{cr}/(EI_2/L^2)$ and L_{eff}/L for $0 \le \alpha \le 50$ and $1 \le \beta \le 50$ are given in Table 6. The plot of $(P_1 + P_2)_{cr}/(EI_2/L^2)$ vs. β is shown in Fig. 7.

The values of L_{eff}/L given above agree with those given in a similar Table in Reference 2, the scope of which Table has been extended here.*

Two special cases can be recognized:

(a) For $\alpha = 0$ and $\beta = 1.0$, only P_1 is applied (at the top) and the column has uniform moment of inertia $I_2 = I_1$:

$$(P_1 + P_2)_{cr} = \text{Euler buckling load} = \pi^2 EI_2/L^2$$
$$L_{eff}/L = 1.0$$

(b) For $\alpha = 1.0$ and $\beta = 1.0$, the column is of uniform cross-section with two equal loads, one applied at top and the other at mid-height:

$$(P_1 + P_2)_{cr} = 13.0720\ (EI_2/L^2)$$
$$L_{eff}/L = 0.868916$$

which agree with the corresponding values for $A/L = 0.50$ previously given in Table 5.

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* *Table 2-6 (p. 100) gives values of L_{eff} for $\alpha = 0, 0.25, 0.50, 0.75, 1.00$ and $\beta = 1.00, 1.25, 1.50, 1.75, 2.00$ and one value for $\alpha = 2.00, \beta = 1.00$.*

Example:

Given: A hinged column with $E = 30{,}000$ kips/in.^{2}, $I_1 = 50$ in.^{4}, $I_2 = 100$ in.^{4}, $L = 100$ in. It carries two loads, P_1 at the top end and P_2 at mid-height such that $P_2 = 2.0\, P_1$. Find the critical values of these loads.

Solution:

$$\alpha = \frac{P_2}{P_1} = 2.0$$

$$\beta = \frac{I_2}{I_1} = 2.0$$

From Table 6 or Fig. 7;

$$(P_1 + P_2)_{cr} = 10.28\left(\frac{EI_2}{L^2}\right) = \frac{10.28 \times 30{,}000 \times 100}{(100)^2}$$
$$= 3{,}084 \text{ kips}$$

$$(3P_1)_{cr} = 3{,}084 \text{ kips or } (P_1)_{cr} = 1{,}028 \text{ kips}$$

$$(P_2)_{cr} = 2.0 \times 1{,}028 = 2{,}056 \text{ kips}$$

Alternatively, $L_{eff}/L = 0.979756$ (Table 6)

$$L_{eff} = 97.98 \text{ in.}$$

$$(P_1 + P_2)_{cr} = \frac{\pi^2 EI_2}{L_{eff}^2} = \frac{9.8696 \times 30{,}000 \times 100}{(97.98)^2}$$
$$= 3{,}084 \text{ kips}$$

$$(P_1)_{cr} = 1{,}028 \text{ kips and } (P_2)_{cr} = 2{,}056 \text{ kips}$$

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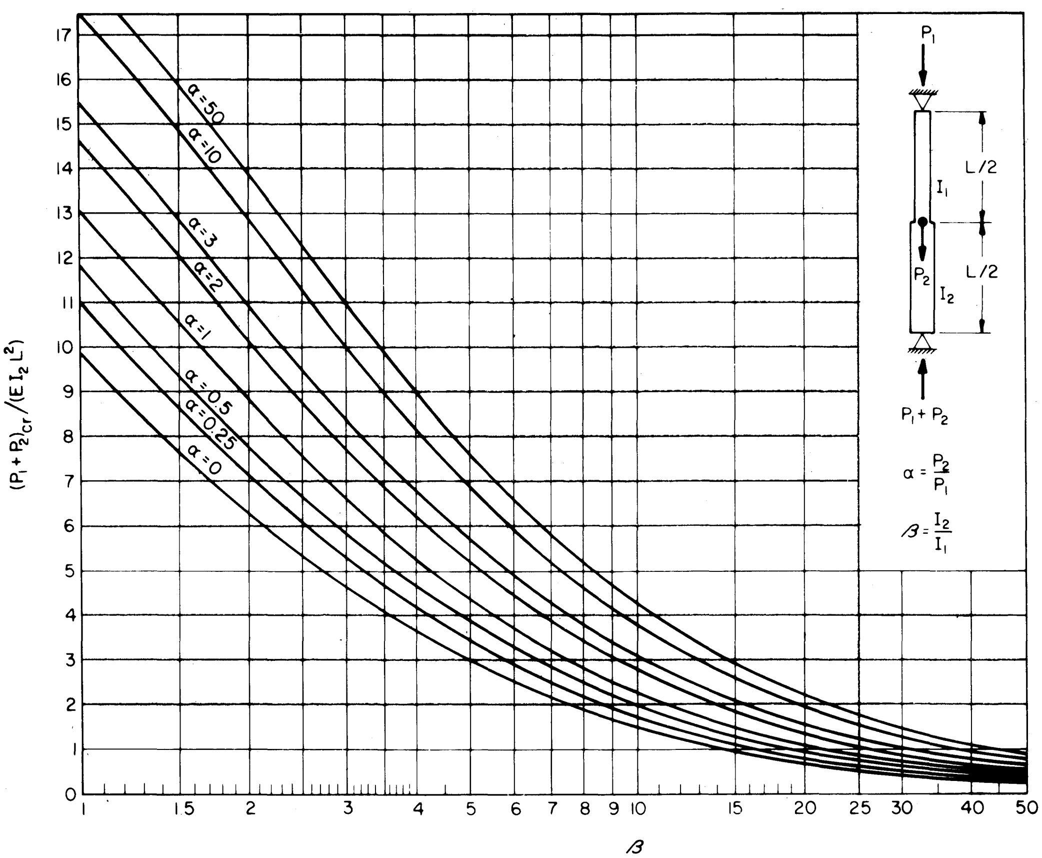

Fig. 7. Unsymmetrically stepped columns with end and intermediate axial loads

Table 6. Critical Buckling Loads for Hinged Non-Prismatic Columns with End Load P_1 at Top and Load P_2 at Mid-Height, I_1 Top Half, I_2 Lower Half

P_2/P_1	I_2/I_1	L_{eff}/L	$(P_1+P_2)_{cr}/(EI_2/L^2)$
0	1.00	1.000000	9.869604
	1.25	1.062289	8.746095
	1.50	1.123540	7.818481
	1.75	1.183209	7.049806
	2.00	1.241077	6.407705
	2.50	1.351329	5.404773
	3.00	1.454745	4.663654
	4.00	1.644268	3.650519
	10.00	2.505679	1.571987
	50.00	5.499653	0.326309
0.25	1.00	0.949037	10.958044
	1.25	1.005055	9.770583
	1.50	1.060455	8.776382
	1.75	1.114673	7.943371
	2.00	1.167446	7.241459
	2.50	1.268391	6.134703
	3.00	1.363424	5.309309
	4.00	1.538155	4.171568
	10.00	2.336207	1.808328
	50.00	5.119076	0.376632
0.50	1.00	0.913974	11.814954
	1.25	0.965533	10.586828
	1.50	1.016755	9.547004
	1.75	1.067074	8.667830
	2.00	1.116199	7.921664
	2.50	1.210478	6.735746
	3.00	1.299515	5.844363
	4.00	1.463685	4.606856
	10.00	2.216692	2.008579
	50.00	4.850063	0.419571
0.75	1.00	0.888392	12.505205
	1.25	0.936617	11.250608
	1.50	0.984703	10.178625
	1.75	1.032085	9.265506
	2.00	1.078460	8.485780
	2.50	1.167713	7.238153
	3.00	1.252230	6.294073
	4.00	1.408450	4.975276
	10.00	2.127670	2.180175
	50.00	4.649274	0.456593
1.00	1.00	0.868916	13.072049
	1.25	0.914555	11.799937
	1.50	0.960197	10.704807
	1.75	1.005286	9.766077
	2.00	1.049511	8.960362
	2.50	1.134830	7.663697
	3.00	1.215810	6.676794
	4.00	1.365816	5.290731
	10.00	2.058699	2.328703
	50.00	4.493432	0.488814
1.50	1.00	0.841255	13.945835
	1.25	0.883143	12.654278
	1.50	0.925222	11.529422
	1.75	0.966957	10.555654
	2.00	1.008030	9.712987
	2.50	1.087580	8.344061
	3.00	1.163372	7.292263
	4.00	1.304267	5.801851
	10.00	1.958668	2.572633
	50.00	4.266907	0.542093
2.00	1.00	0.822572	14.586535
	1.25	0.861874	13.286550
	1.50	0.901483	12.144632
	1.75	0.940884	11.148792
	2.00	0.979756	10.281679
	2.50	1.055272	8.862807
	3.00	1.127434	7.764572
	4.00	1.261959	6.197398
	10.00	1.889542	2.764309
	50.00	4.109967	0.584283
2.50	1.00	0.809119	15.075603
	1.25	0.846534	13.772451
	1.50	0.884329	12.620363
	1.75	0.922010	11.609892
	2.00	0.959260	10.725738
	2.50	1.031793	9.270741
	3.00	1.101270	8.137891
	4.00	1.231084	6.512149
	10.00	1.838882	2.918717
	50.00	3.994717	0.618483
3.00	1.00	0.798977	15.460785
	1.25	0.834952	14.157164
	1.50	0.871362	12.998792
	1.75	0.907724	11.978209
	2.00	0.943727	11.081724
	2.50	1.013965	9.599609
	3.00	1.081374	8.440099
	4.00	1.207560	6.768340
	10.00	1.800149	3.045670
	50.00	3.906452	0.646748
10.00	1.00	0.753370	17.389318
	1.25	0.782726	16.109433
	1.50	0.812687	14.943512
	1.75	0.842868	13.892494
	2.00	0.872989	12.950374
	2.50	0.932347	11.353877
	3.00	0.989918	10.071657
	4.00	1.098832	8.174054
	10.00	1.619315	3.763889
	50.00	3.492379	0.809202
50.00	1.00	0.732808	18.378909
	1.25	0.759106	17.127522
	1.50	0.786047	15.973586
	1.75	0.813298	14.921070
	2.00	0.840606	13.967390
	2.50	0.894714	12.329088
	3.00	0.947508	10.993442
	4.00	1.048012	8.986014
	10.00	1.533546	4.196684
	50.00	3.294608	0.909269