Automated Analysis of Grid Beam Systems

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THE SOLUTIONS of grid systems have been handled by many authors and are too numerous for all of them to be mentioned in this paper. However, they generally approach the problem using a few standard procedures.[1] These methods can be divided into two main categories—flexibility methods and stiffness methods.

The flexibility methods are usually exact methods which use the inter-reactive forces between the two orthogonal grid systems as the redundants to the system. The equations of consistent deflection are then determined and solved for the redundant values. The stiffness methods are in general more variable with respect to method and assumptions. The exact methods can be divided into those using the actual existing grid system and those methods substituting an equivalent plate for the grid system. Using the existing grid system for the solution, there are several methods which can be used to solve the problem.

One method is to solve the slope deflection equations for the grid system. This requires two equilibrium equations of moment in each orthogonal direction, plus one equilibrium equation for the external force. These equations are then solved simultaneously for the deflections and rotations at each node point of the grid, which then in turn are used to determine the moments and shears in the individual beam elements.

A second method is the reaction distribution method.[2] In this method a stiffness matrix is determined on the basis of the restraining forces at all nodes when a unit deflection is permitted at a single node. This requires the intermediate solution of the continuous beams that are deflected for the particular deflected node. The equations of equilibrium, in terms of the external loads, can then be solved simultaneously for the deflections, and the moments or shears, or a reaction distribution process can be employed as in the referenced paper.

A third method is to apply finite difference techniques to the differential equation of equilibrium for continuous beams, using the grid node points as the finite difference grid. This, of course, leads to an inexact solution in which the error will depend on the variation of the load. Another method would be to substitute an equivalent plate for the grid system. This idealized plate would then be analyzed using the differential equation of equilibrium for flat plates.[3]

All of the above methods are usable and lead to either exact solutions or reasonable solutions. This paper will introduce a method which will lead to an exact solution using Clapeyron's three-moment equation,[4] which can be derived from the slope deflection equations. The slope deflection equations require essentially three unknown quantities per node point; the proposed method requires only two unknown quantities per node point for the general solution. The method is also ideally suited for automated procedures using the digital computer.

PRINCIPLES OF PROPOSED METHOD OF ANALYSIS

Consider two beam elements (Fig. 1) taken from one of a series of continuous beams in the x-direction that are used to create an orthogonal grid system (Fig. 2). The spacing between the beams in both directions is constant but not necessarily equal. Using the three moment equation, it is possible to express the relationship between the moments at the node points (**W**, **C** and **E**) of the beam, and the deflections at the node points (**W**, **C** and **E**) as follows:

$$M^x{}_w + 4M^x{}_C + M^x{}_E = \frac{6EI}{h^2}(-w_W + 2w_C - w_E) \quad (1)$$

where M^x is the moment in the beam at the indicated node point (compression on the top fiber considered positive) and w the deflection of the beam at the indicated node point (downward deflection considered positive). Applying equilibrium to the node point, the relationship between the load on the node point and the moments can be determined as follows:

$$-M^x{}_W + 2M^x{}_C - M^x{}_E = P^x{}_C h \quad (2)$$

where $P^x{}_C$ is the reacting load on the beam at the node point **C**, and h is the length of the beam between node points in the x-direction. Adding Equations (1) and (2), the result expresses the moment at a node point in terms of the second difference of the deflections near the node, and the reacting load as follows:

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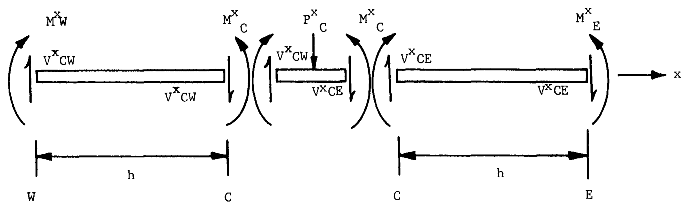


Fig. 1. Two beam elements of continuous beam in the x-direction

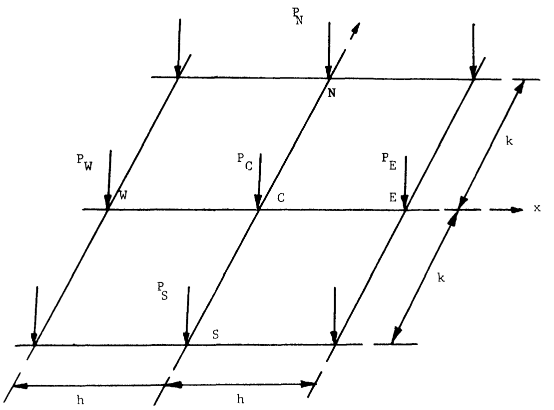


Fig. 2. Typical layout of part of an orthogonal grid system

$$6M^x{}_C = \frac{6EI}{h^2}(-w_W + 2w_C - w_E) + P^x{}_C h \qquad (3)$$

For computational purposes, Equations (2) and (3) are divided by h and multiplied by 6. The equations then use the moment in terms of the variable $6M^x/h$ and the equations appear in the following form:

$$-\frac{6M^x{}_W}{h} + 2\frac{6M^x{}_C}{h} - \frac{6M^x{}_E}{h} = 6P^x{}_C \qquad (2)$$

$$\frac{6M^x{}_C}{h} = K_x(-w_W + 2w_C - w_E) + P^x{}_C \qquad (3)$$

where $K_x = 6EI/h^3$.

Now consider two beam elements taken from one of a series of continuous beams in the y-direction that are used to create the orthogonal grid system. The length of the beam between node points in the y-direction will be given the constant k. From the results of the preceding development, Equations (2) and (3) can now be written for continuous beams in the y-direction as follows:

$$-\frac{6M^y{}_S}{k} + 2\frac{6M^y{}_C}{k} - \frac{6M^y{}_N}{k} = 6P^y{}_C \qquad (4)$$

$$\frac{6M^y{}_C}{k} = K_y(-w_S + 2w_C - w_n) + P^y{}_C \qquad (5)$$

where $K_y = 6EI/k^3$.

Next, consider the equilibrium of a node point **C** of the grid system. It can be seen that the sum of the reactive forces on the orthogonal beams that intersect at that node point must be equal to the externally applied force at that point. Thus,

$$P^x{}_C + P^y{}_C = P_C \qquad (6)$$

where P_C is the external load at the node point **C**. Equations (2) through (6) now form the complete set of equations necessary for the solution of the problem. However, substituting Equation (3) into Equation (2) and Equation (5) into Equation (4), the moments at the node points will be eliminated as unknowns from the necessary equations and can be found later by back substitution into Equations (3) and (5) when the deflections and reactive forces are determined. Equation (6) can also be used to eliminate either $P^x{}_C$ or $P^y{}_C$ from these reduced equations, so that now only two equations remain which are necessary for the solution of the two remaining unknowns, namely, the deflection of the node point and one of the reactive forces. The two remaining equations written using the standard difference operators would appear in the following form:

$$-\nabla^2 x(-K_x\nabla^2 x w_C + P^x{}_C) - 6P^x{}_C = 0 \qquad (7)$$

$$-\nabla^2 y(-K_y\nabla^2{}_y w_C + P_C - P^x{}_C) + 6P^x = 6P_C \qquad (8)$$

The reactive force $P^y{}_C$ can be obtained upon completion of the solution for the deflections and $P^x{}_C$ reactive forces by back substitution into Equation (6).

BOUNDARY CONDITIONS

There are two types of boundary conditions that are of general interest with respect to this type of problem. The first type is the free boundary, where supporting beams are used to support the interior members. The second type occurs when the grid system is continuous over interior supports.

In the first type the moment at the end of the beam being carried by the spandrel beam will be assumed to be zero (since the torsional stiffness of the beams is ignored) and, when expressed in terms of Equation (2) or (4), will result in the proper equivalent expression for Equation (7) or (8). The moments and reactive forces for points exterior to the grid system will also be assumed to be zero.

In the second type of boundary condition, where the node point rests on an interior support, the external load (in this case the reaction) is in general an unknown and, therefore, Equation (8) must be rewritten using $P^y{}_C$ instead of P_C. The two equations are still sufficient for the solution to the problem, since the deflection is now equal to zero and there are only two unknowns remaining, namely, the two reactive forces.

PROBLEM ORGANIZATION

One of the most time consuming phases in the solution of any indeterminate structural problem is the creation of the matrices needed for the solution. The aim of this computer program is to eliminate this as completely as possible, yet still permit as much flexibility as possible.

Thus, the problem must be phrased so that the input to the computer is in the simplest possible form. In essence there are three main phases. The first phase concerns the numbering of the nodes of the grid. The second phase concerns the identification of the node numbers associated with each unknown moment in the grid. The third phase concerns the identification of node numbers associated with the moments used in the equilibrium Equations (7) and (8). A more detailed explanation of these phases can be outlined as follows:

1. Node Identification Number:
    a. Number the deflecting nodes from **1** to **N1**.
    b. Number the reaction nodes which support one or more continuous beams from (**N1** + **1**) to **N2**.
    c. Number the reaction nodes which support the corners of the grid if they exist from (**N2** + **1**) to **N3**. (These need not be listed unless an output of the continuous beam reaction shears is desired.)
2. Moment Equations:
    a. Determine the node number of the deflecting nodes associated with the determination of the moments M^x in consecutive order, i.e., **W**, **C**, and **E**, for each node of the grid.
    b. Repeat the same process for M^y, i.e., **S**, **C**, and **N**.
    c. If the associated node is nonexistent or has zero deflection, a zero is substituted for the node number. If the moment is equal to zero, zeroes are substituted for all node numbers.
3. Equilibrium Equations:
    a. Determine the node numbers of moments associated with the equilibrium equation of the x-direction beam (**W**, **C**, and **E**) for each grid node.
    b. Repeat the process for the y-direction beam.
    c. If the associated node is nonexistent, a zero is substituted for the node number.

COMPUTER PROGRAM

The program is written essentially to accept information in the form as determined in Phases 2 and 3 above, then create Equations (2) through (6), and subsequently use these equations to create Equations (7) and (8). This is accomplished in the following manner:

After the preliminary information has been input, consisting of the values of h and k as well as the numbering information obtained from Phase 1, the information from Phase 2 will be input. Supplementary data containing the external load and the relative stiffness of the beams at the node points will be input along with the moment equations for each node. The program will then create the coefficient matrix which will relate the node moments to the node deflections w and the node reactive forces P^x. The information from Phase 3 is then input. As the equilibrium equations for each node are input, the program will create the coefficient matrix relating the external loading to the node unknowns, using the matrix relating the node moments to these unknowns. Thus Equations (7) and (8) are formed. To form a strong diagonal in this coefficient matrix, Equations (7) and (8) are added together and substituted for Equation (8). This equation will create a strong diagonal coefficient for the deflections w. For interior support nodes, this step is omitted. When all of the information from Phase 3 has been input, the matrix is inverted[5] and the unknown deflections and reactive forces P^x determined. These values can then be used to determine the node moments. The program will print out the node moments, reactive forces, external loading, and relative deflection. A detailed flow chart of the program appears in Appendix A.

EXAMPLE

A simple problem that will illustrate the use of this method of analysis would be a uniformly loaded grid roof system supported at the corners. The roof will be square and will be divided into four equal bays in each direction (Fig. 3). For the preliminary analysis, the moment of inertia of all roof beams (or trusses) will be considered identical. When this preliminary analysis is completed, the members can be sized, and the moments of inertia of these members may be used in another analysis to optimize the design of the structure. For the general analysis, the distance between beams will be considered unity, and the concentrated load at each node will be expressed in terms of an interior concentrated load considered as unity. Using these assumptions, the output loads will be the actual loads divided by P, where P is a typical interior node load, and the output moments will be the actual moments divided by Ph, where h is the actual distance between beams.

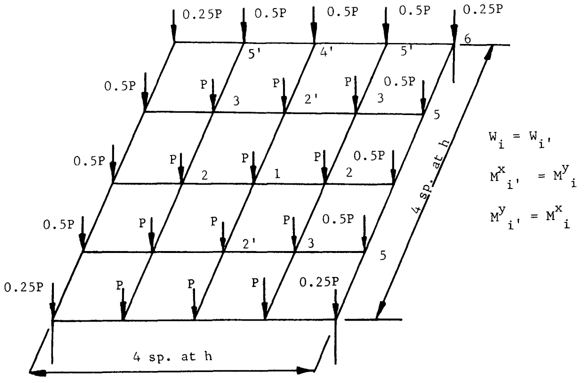


Fig. 3. Roof grid system for example problem

Computer Input and Output:

Input:

h	k	**N1**	**N2**	**N3**
1.00	1.00	5	5	6

	M_x				M_y				
Node	**W**	**C**	**E**	K_x	**S**	**C**	**N**	K_y	Load
1	2	1	2	1.00	2	1	2	1.00	1.000
2	1	2	4	1.00	3	2	3	1.00	1.000
3	2	3	5	1.00	2	3	5	1.00	1.000
4	0	0	0	1.00	5	4	5	1.00	0.500
5	0	0	0	1.00	4	5	0	1.00	0.500
6	0	0	0	1.00	0	0	0	1.00	0.250

	EQ X			EQ Y		
Node	**W**	**C**	**E**	**S**	**C**	**N**
1	1 2	1 1	1 2	1 2	2 1	1 2
2	1 1	1 2	1 4	2 3	2 2	2 3
3	2 2	1 3	1 5	2 2	2 3	1 5
4	1 2	1 4	0 0	2 5	2 4	2 5
5	1 3	1 5	0 0	2 4	2 5	0 0
6	2 5	0 0	0 0	2 5	0 0	0 0

Note: ($x = 1, y = 2$)

Output:

Node	M_x	M_y	P_x	P_y	P	$PSUM$	W
1	0.7573	0.7573	0.5000	0.5000	1.000	0.000	29.97
2	0.5074	1.2426	0.2574	0.7426	1.000	0.000	27.95
3	0.8713	0.8713	0.5000	0.5000	1.000	0.000	24.60
4	0.0000	2.3787	−0.5074	1.0073	0.500	0.000	23.14
5	0.0000	1.8750	−0.8713	1.3713	0.500	0.000	16.51
6	0.0000	0.0000	−1.8750	−1.8750	0.250	−4.000	0.00

CONCLUSIONS

The method of analysis outlined in this paper produces an exact solution to a beam grid system. The computer program written to accommodate this method was formulated so that the input of a specific problem can be determined quickly and efficiently.

Although the method of analysis was developed especially for beam type members such as WF sections of plate girders, two-way truss analysis is also applicable where the upper and lower chords are of continuous section and the shearing deformation of the trusses are considered negligible.

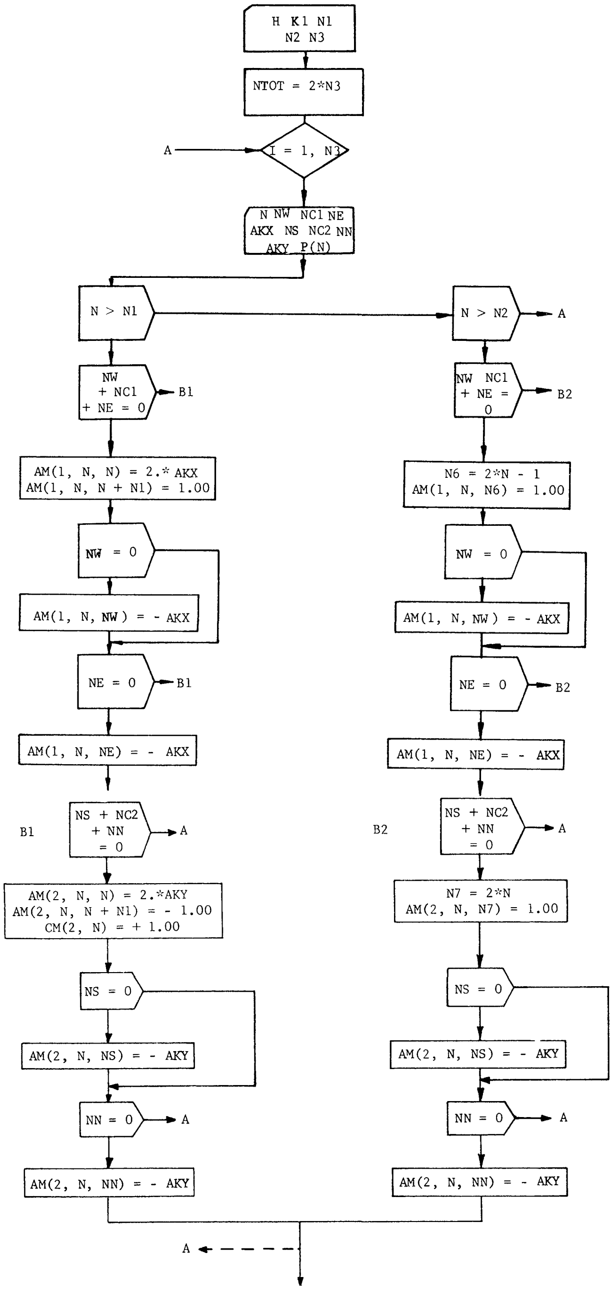
Certain extensions of this computer program also seem feasible. One is the incorporation of a design subroutine (using either WF, plate girder, or truss type members) which cycles the analysis for optimization of the grid members. However, the author feels that such refinements of the computer program should be left in the hands of the design engineer.

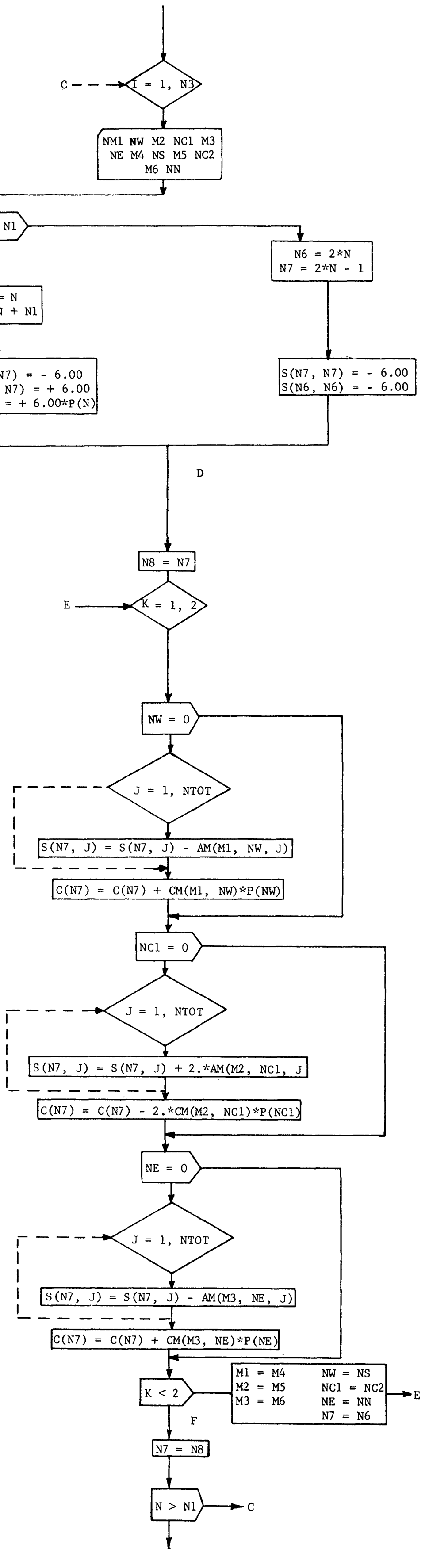
REFERENCES

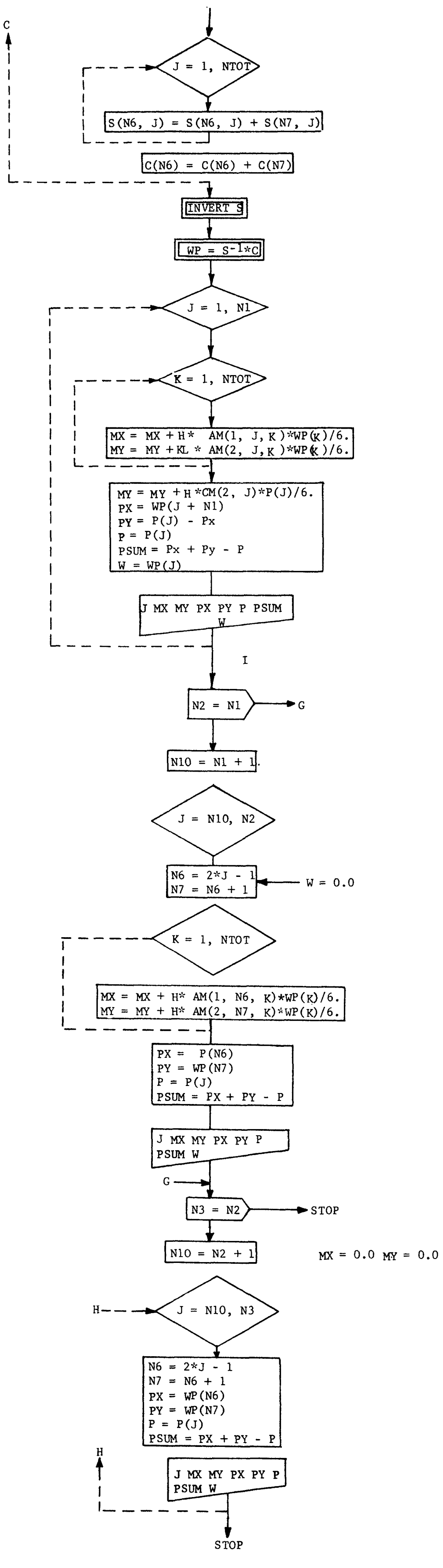
1. *American Institute of Steel Construction* Design Examples, Space Forms in Steel (revised May, 1966).
2. *Fader, I.* Grid Analysis by the Reaction Distribution Method *ASCE Trans., Vol. 128, 1963, Part II, pp. 255 to 283.*
3. *Timoshenko, S. and Woinowskv-Krieger, S.* 'Theory of Plates and Shells *McGraw-Hill Book Co., Inc., 1959.*
4. *Asplund, S. O.* Structural Mechanics: Classical and Matrix Methods *Prentice-Hall 1966, pp. 360. (For historical development of Clapeyron's three moment equation.)*
5. *James, M. L., Smith, G. M., and Wolford, J. C.* Analog and Digital Computer Methods in Engineering Analysis *International Textbook Co., 1964.*

APPENDIX A

Flowchart of Computer Program




C
I = 1, N3
NM1 NW M2 NC1 M3
NE M4 NS M5 NC2
M6 NN
N > N1
N6 = 2*N
N7 = 2*N - 1
N6 = N
N7 = N + N1
S(N7 N7) = - 6.00
S(N6, N7) = + 6.00
C(N6) = + 6.00*P(N)
S(N7, N7) = - 6.00
S(N6, N6) = - 6.00
D
N8 = N7
E
K = 1, 2
NW = 0
J = 1, NTOT
S(N7, J) = S(N7, J) - AM(M1, NW, J)
C(N7) = C(N7) + CM(M1, NW)*P(NW)
NC1 = 0
J = 1, NTOT
S(N7, J) = S(N7, J) + 2.*AM(M2, NC1, J
C(N7) = C(N7) - 2.*CM(M2, NC1)*P(NC1)
NE = 0
J = 1, NTOT
S(N7, J) = S(N7, J) - AM(M3, NE, J)
C(N7) = C(N7) + CM(M3, NE)*P(NE)
K < 2
M1 = M4
M2 = M5
M3 = M6
NW = NS
NC1 = NC2
NE = NN
N7 = N6
E
F
N7 = N8
N > N1
C


C
J = 1, NTOT
S(N6, J) = S(N6, J) + S(N7, J)
C(N6) = C(N6) + C(N7)
INVERT S
WP = S^{-1}*C
J = 1, N1
K = 1, NTOT
MX = MX + H* AM(1, J, K)*WP(K)/6.
MY = MY + KL * AM(2, J, K)*WP(K)/6.
MY = MY + H*CM(2, J)*P(J)/6.
PX = WP(J + N1)
PY = P(J) - Px
P = P(J)
PSUM = Px + Py - P
W = WP(J)
J MX MY PX PY P PSUM
W
I
N2 = N1
G
N10 = N1 + 1
J = N10, N2
N6 = 2*J - 1
N7 = N6 + 1
W = 0.0
K = 1, NTOT
MX = MX + H* AM(1, N6, K)*WP(K)/6.
MY = MY + H* AM(2, N7, K)*WP(K)/6.
PX = P(N6)
PY = WP(N7)
P = P(J)
PSUM = PX + PY - P
J MX MY PX PY P
PSUM W
G
N3 = N2
STOP
N10 = N2 + 1
MX = 0.0 MY = 0.0
H
J = N10, N3
N6 = 2*J - 1
N7 = N6 + 1
PX = WP(N6)
PY = WP(N7)
P = P(J)
PSUM = PX + PY - P
H
J MX MY PX PY P
PSUM W
STOP