# Deformation Analysis of Structures Near Collapse Load

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THE PRIMARY FUNCTION of any structure is to provide adequate strength to resist the loads acting on it without undergoing excessive deformations which, in turn, might render the structure of no further use. For every structure, therefore, deformations under working loads must be within certain definite limits. If the loads are increased beyond their working values, the material at the critical sections is expected to yield. This brings about a sharp increase in the deformations. These increased deformations may cause several forms of premature local buckling in structures composed of mild steel, while in reinforced concrete structures, local fracture of concrete, with resulting failure of the section, is likely to occur due to the brittle nature of the concrete material. These effects may lead to local collapse of structures at loads lower than those otherwise expected to produce failure. It is imperative, therefore, to develop methods for estimating hinge rotations and other deformations at ultimate loads which are simple and economically feasible to put into practice by an average practicing engineer. Various methods[1, 2, 6] to accomplish this have been proposed for steel as well as reinforced concrete structures. They are primarily based on either slope deflection equations or energy methods. The slope deflection equations[2, 6] used for estimating deformations of any structure at ultimate loads, if extended as shown later in this paper, lend themselves to the introduction of the principle of elastic moment distribution. The method presented herein makes use of this principle.

The following assumptions have been made in the development of the proposed method:

1. The moment curvature relationship is idealized as shown in Fig. 1.
2. The plastic hinges are concentrated, i.e., the spread of plastic zone is neglected.
3. Regions of members between plastic hinges remain linearly elastic.
4. Deformations are small enough so that $\tan \theta = \theta$, where $\theta$ = rotation.
5. The material is homogeneous.

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### DEVELOPMENT OF THE TECHNIQUE

When the loads carried by a structure are increased to near their ultimate values, the material at the sections carrying maximum stresses may be assumed to yield. Any increase in load beyond this stage brings about a rapid increase in curvature of the members at these sections. The increased curvatures extend over some length of member, forming so-called "plastic hinges" which behave as if they were rusty hinges. The hinges which form at loads lower than the collapse load must have sufficient rotation capacity to permit the structure to form all the requisite hinges necessary to convert it into a mechanism. Otherwise, local failure will occur at lower loads. Just before the collapse, it is implied that all the hinges except the one which forms last must have undergone rotations. The magnitude of these rotations will be unknown, but must be found, to ensure that none will be greater than the rotation capacity of the member in which it occurs. Once the hinge which forms last is

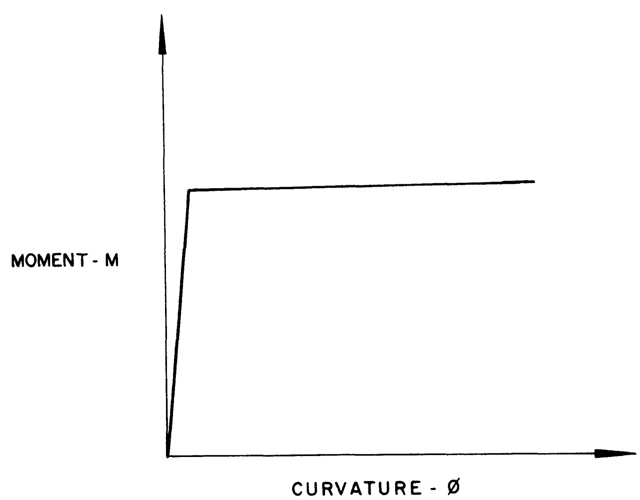



*Fig. 1. Idealized Moment-Curvature relationship*

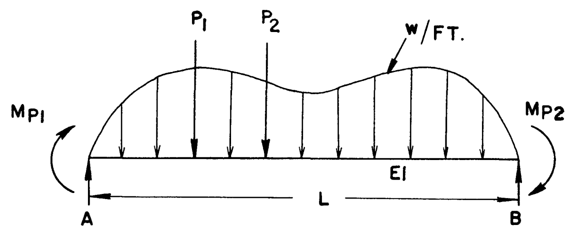



*Fig. 2. Free body diagram of a typical segment*

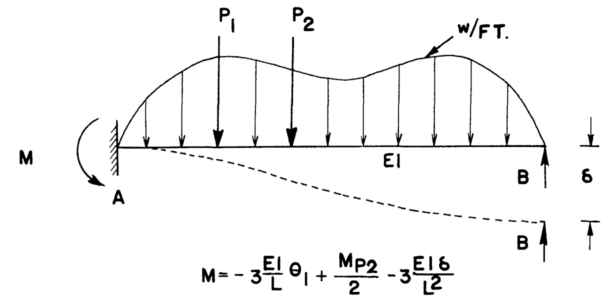



*Fig. 3. Artificial conditions created at* **A** *and* **B**

located, the rotations of all hinges can be calculated. The last hinge can be located by using the Maximum Displacement Theorem,[5] according to which each hinge in turn can be assumed to have formed last and the corresponding deflections at a particular section determined. The trial hinge that gives the greatest value of the deflection at the section under consideration is the one which forms last. The technique is employed each time for computing the deflection of the structure and subsequently the rotations of the hinges.

Figure 2 shows the free-body diagram of a typical segment **AB** of the structure under consideration. Both ends of **AB** are acted upon by moments of arbitrary magnitude which may or may not be equal to the plastic moment capacity of the sections at **A** or **B**. In this case the end **A** is assumed to be a plastic hinge. It is also assumed that the end **B** moves vertically downward by an amount $\delta$ to be ascertained. Assuming the clockwise rotation of the member as positive and the net rotation of end **A** to be counterclockwise, one obtains

$$-\theta_A = \theta_1 + \frac{M_{p1}L}{3EI} - \frac{M_{p2}L}{6EI} + \frac{\delta}{L} \tag{1}$$

where $\theta_A$ = Net rotation of end **A**

$\theta_1$ = Rotation of end **A** due to external loads considering segment **AB** as simply supported.

$M_{p1}$ = Plastic moment capacity of the section at **A**.

$M_{p2}$ = Resisting moment at **B**.

$\delta$ = Vertical displacement of end **B** with respect to end **A**.

$L$ = Length of **AB**.

$EI$ = Flexural rigidity of the member.

Simplifying Equation (1),

$$\frac{3EI}{L}\theta_A + M_{p1} = \frac{M_{p2}}{2} - \frac{3EI}{L}\theta_1 - \frac{3EI\delta}{L^2} \tag{2}$$

If the hinge at **A** is assumed to have formed last, it will not undergo any rotation. Equation (2) then reduces to

$$M_{p1} = \frac{M_{p2}}{2} - \frac{3EI}{L}\theta_1 - \frac{3EI\delta}{L^2} \tag{3}$$

On the other hand, hinge **A**, when assumed to have formed last, fulfills all the necessary conditions of a fixed support. One can, therefore, fix end **A** and evaluate the subsequent effects due to fixity as shown in Fig. 3, without considering the actual moments acting there. The effects of fixing **A** are:

1. The fixed end moment $3EI_{\theta_1}/L$ is produced due to the external loads, where $3EI/L$ is the stiffness of segment **AB** taken as simply supported.
2. Half of the moment acting at **B** is carried over to **A**.
3. A fixed end moment $3EI\delta/L^2$ is produced due to the settlement of support **B**.

Assuming a clockwise moment on the member as positive, the total moment produced at **A** is then equal to

$$M = \frac{-3EI}{L}\theta_1 + \frac{M_{p2}}{2} - \frac{3EI\delta}{L^2} \tag{4}$$

This moment was produced by creating *artificial conditions*, such as fixing the end at which the hinge is assumed to have formed last and then deflecting the other end in the direction indicated by the deflected shape of the structure. It is, therefore, designated as the *artificial moment*. Of course no moment except the actual moment can exist at **A**. This actual moment is $M_{p1}$ acting in the clockwise direction and hence positive. The artificial moment in Equation (4), when equated to the actual moment, yields an expression identical to Equation (3):

$$M = M_{p1} = \frac{M_{p2}}{2} - \frac{3EI}{L}\theta_1 - \frac{3EI\delta}{L^2} \tag{5}$$

Equation (5) can then be solved for $\delta$, the only unknown.

If two members terminate at hinge **A**, the fixed end moments produced due to the external loads and those due to the settlements of the ends (the direction of the settlement being indicated by the deflected shape of the structure) can be evaluated. Since end **A** is treated as a rigid support, continuity exists there and hence the moments evaluated above, if unbalanced, can be distributed between the members meeting at **A** in proportion to their stiffnesses. The final moments after distribution are designated as artificial moments of the respective members. In the case of frames, it is necessary to establish a relationship between the horizontal sway of the vertical member and the vertical deflection of the horizontal segment. This relationship can be established without great difficulty by creating artificial conditions at any rigid joint at which two members, one moving vertically and the other laterally under the action of the external loads, terminate. In either of these cases, the artificial moment acting on any member, when equated to the actual moment acting there, gives the unknown deflection of that member. All hinges are treated as above. The trial hinge which gives greatest deflection at any assumed section is the one which forms last. The rotation of this hinge is then evidently zero prior to collapse.

Once the deflections of various segments of the structure are determined, rotations of all hinges can be obtained with relative ease. The artificial conditions are created at the hinge at which the rotation is to be calculated. Since continuity no longer exists at the hinge under consideration, the artificial moments thus produced can not be distributed. As is apparent from Equation (2), the rotation of the member can be obtained by equating the artificial moment acting on any member to the actual moment acting there plus the product of the stiffness of the member taken as simply supported and the unknown rotation to be evaluated. If two or more members meet at a hinge, the rotation or the angle of discontinuity produced there is the algebraic sum of the discontinuities produced by the rotations of the members terminating there.

The evaluation of the deflections of various segments of the structure and, subsequently, the rotations of the plastic hinges, by making use of the collapse mechanism as explained above, becomes highly involved as the indeterminacy of the structure increases. In this situation, as explained by Professor Baker,[1] it is advisable to assume a mechanism such that the structure becomes determinate at ultimate loads. The plastic hinges can be assumed to occur at sections at which the elastic moments are likely to be maximum. Arbitrary moments are then assumed to act at all the plastic hinge locations. The rotations of these hinges can be estimated then by using the method described above. For the hinges to occur at those sections, these rotations must have the same signs as those of the moments acting there. If the calculated rotations are not of the correct sign or are in excess of the allowable rotations, either the assumed values of the moments at the hinges or the locations of the hinges, or both, will have to be modified until all the rotations are of the correct sign and lie within the permissible value.

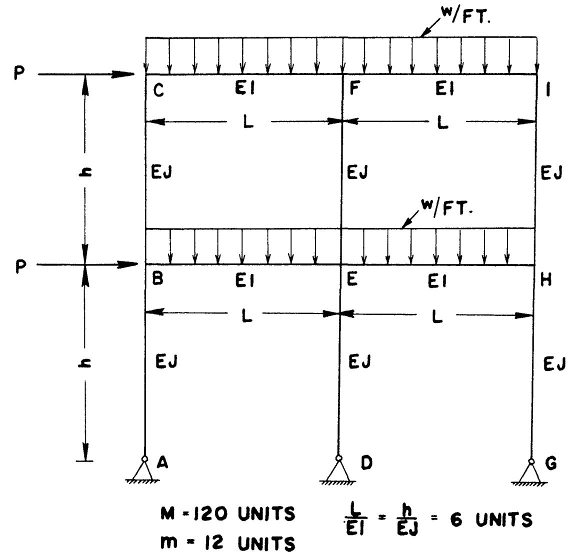



*Fig. 4. Illustrative example*

**ILLUSTRATIVE EXAMPLE**

The rotations of plastic hinges of a two bay, two story frame, shown in Fig. 4, are calculated by the proposed method and then verified by comparison to those obtained by Baker.[1] As mentioned earlier, initial steps, such as locating plastic hinges and assuming arbitrary moments to act at the assumed hinging sections, are exactly similar to those used by Baker and hence will not be discussed here. To avoid confusion, the plastic hinges are also designated by the same notations as those used by Baker. The bending moment diagrams produced by the arbitrary moments assumed to act at the assumed hinging sections and the bending moment diagrams due to the external loads can be superimposed to obtain the final assumed bending moment diagram as shown in Fig. 5. The free body diagram of the entire frame is shown in Fig. 6.

Prior to determining the rotations of the hinges, it is necessary to evaluate the displacements of the ends of the members from their original positions. Since the ends of the beams remain at the same level, one has to determine only the lateral movement of the vertical members of the top as well as the bottom story. Once the deflections are known, the rotation of any hinge can be obtained as the algebraic sum of the rotations of the members meeting there.

**Sway of the Top Story**—Since the horizontal displacements of points **C, F** and **I** are equal, and since continuity exists at **C** only, the sway of the top story relative to the bottom story can be obtained by creating artificial conditions at **C.** In Fig. 7a are shown the free body diagrams of members **CB** and **CF** meeting at **C.** As

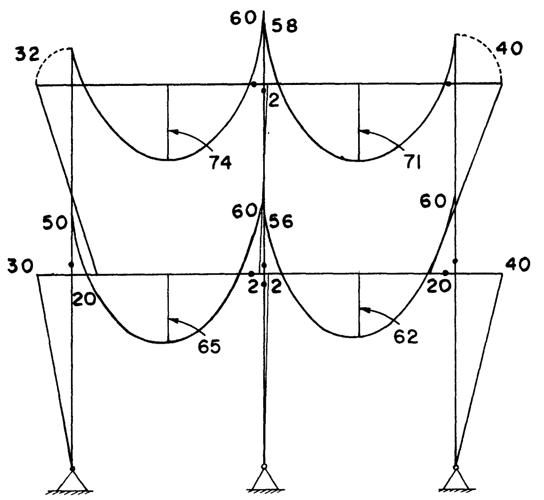



*Fig. 5. Final assumed bending moment diagram*

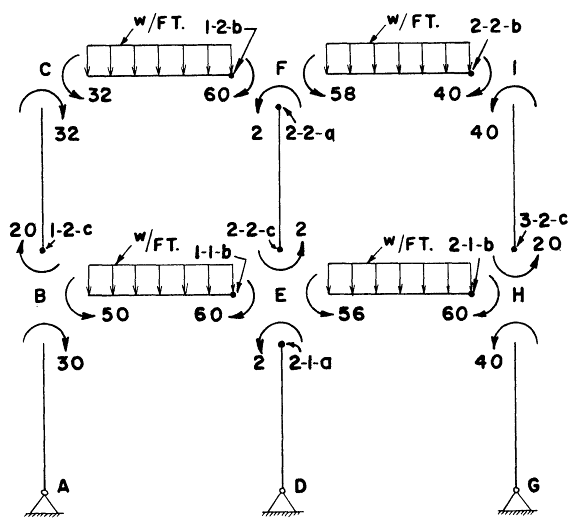
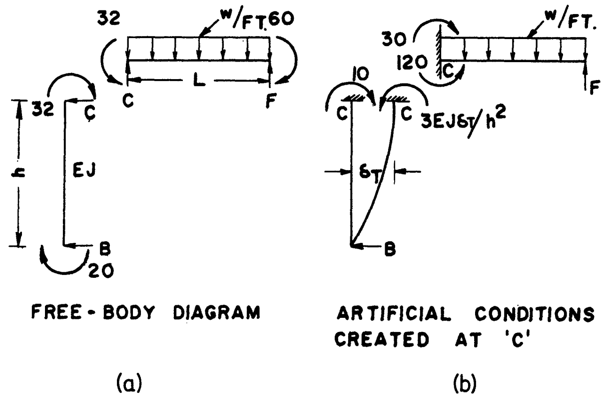



*Fig. 6. Free body diagram of the entire frame*




| CB D.F.=1/2 | CF D.F.=1/2 | |
|---|---|---|
| $10 - 3EJ\delta_T/h^2$ | $-90$ | ............. F.E.M. |
| $40 + 3EJ\delta_T/2h^2$ | $40 + 3EJ\delta_T/2h^2$ | .... DIST. |
| $50 - 3EJ\delta_T/2h^2$ | $-50 + 3EJ\delta_T/2h^2$ | |

MOMENT DISTRIBUTION

ARTIFICIAL MOMENT @ 'C' ON 'CB' = $50 - 3EJ\delta_T/2h^2$

ACTUAL MOMENT @ 'C' ON 'CB' = 32 UNITS

EQUATING BOTH,

$$50 - 3EJ\delta_T/2h^2 = 32$$

$$\therefore \delta_T = 12h^2/EJ \quad \text{.............. a}$$

*Fig. 7. Calculations for the deflection of the top story*

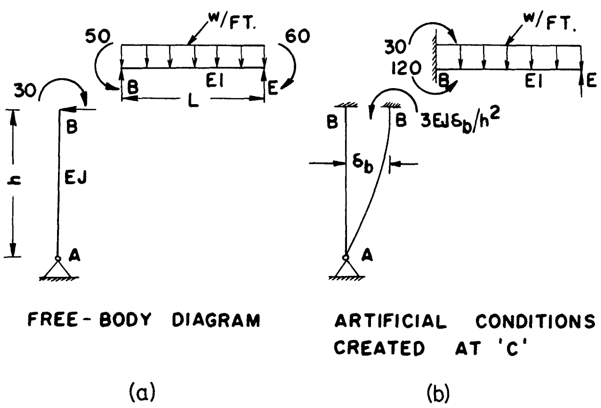
shown in Fig. 7b, artificial conditions, such as fixing **C** and then deflecting **C** in member **CB** by an unknown amount $\delta_T$ to be ascertained, can be created at **C** and their effects determined without considering the actual moments acting there. The moments acting on either side of **C** are unbalanced and hence must be distributed between **CB** and **CF** in proportion to their stiffnesses. The distribution factors can be obtained in a manner similar to that used in an elastic analysis. The moment after distribution is the artificial moment. The artificial moment acting on **CB,** when equated to the actual moment of 32 units acting clockwise on it, gives the deflection, $\delta_T$, of the top story. See Equation "a" in Fig. 7.

**Sway of the Bottom Story**—In the bottom story, since continuity exists between members **BE** and **BA** at joint **B,** the unbalanced moment acting there can be distributed between them. The moment acting on segment **BC** does not have any influence on this distribution. The moment acting on **BA** and **BE** after distribution is 40 units. This distributed moment can be considered as the actual moment acting on these members so far as the calculation of the deflection is concerned. On the other hand, the rotation of hinge **1-2-c** is determined on the basis of the undistributed moment because continuity does not exist either between **BE** and **BC** or between **BA** and **BC.** Figure 8a shows the free body diagrams of members **BA** and **BE** meeting at **B** and Fig. 8b shows the effects of the artificial conditions created at **B.** The unbalanced moment at **B** is distributed between **BA** and **BE** to obtain the artificial moment which, when equated to the actual moment of 40 units acting clockwise on **BA,** yields the value of the sway of the bottom story.

**Rotations of Hinges**—The procedure for the calculation of the rotations of plastic hinges is explained in detail for two of the hinges, **1-2-b** and **2-2-c.** Calculations for the other hinges are shown in Table 1.

**Table 1. Rotations of Plastic Hinges in the Illustrative Example (First Trial)**

| Hinge | 1-2-b | | 2-2-b | | 1-1-b | | 2-1-b | | 2-2-a | | 1-2-c | | 2-2-c | | 3-2-c | | 2-1-a | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Member | **FC** | **FI** | **IF** | **IH** | **EB** | **EH** | **HE** | **HG** | **FE** | **FI** | **BC** | **BE & BA** | **EF** | **EH** | **HI** | **HG** | **ED** | **EH** |
| Rotation | 88 | −84 | 102 | −12 | 70 | −68 | 64 | 60 | −70 | −84 | −80 | −80 | −70 | −68 | −72 | 60 | −16 | −68 |
| Magnitude of net rotation | 172 | | 114 | | 138 | | 4 | | 14 | | 0 | | 2 | | 132 | | 52 | |




| BA D.F.=1/2 | BE D.F.=1/2 | |
|---|---|---|
| $-3EJ\delta_b/h^2$ | $-90$ | F.E.M. |
| $45+3EJ\delta_b/2h^2$ | $45+3EJ\delta_b/2h^2$ | DIST. |
| $45-3EJ\delta_b/2h^2$ | $-45+3EJ\delta_b/2h^2$ | |

MOMENT DISTRIBUTION

ARTIFICIAL MOMENT AT 'B' ON 'BA' = $45 - 3EJ\delta_b/2h^2$

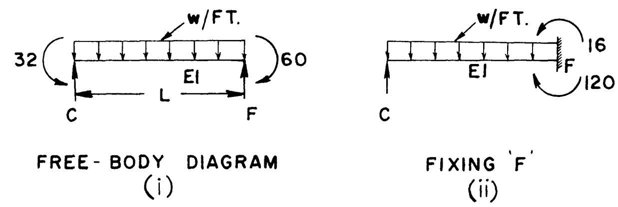
ACTUAL MOMENT AT 'B' ON 'BA' = 40 UNITS

EQUATING BOTH,

$$45 - 3EJ\delta_b/2h^2 = 40$$

$$\therefore \delta_b = 10h^2/3EJ \quad \ldots\ldots\ldots\ldots \text{b}$$

*Fig. 8. Calculations for the deflection of the bottom story*




ARTIFICIAL MOMENT AT 'F' = 104 UNITS

ACTUAL MOMENT AT 'F' = 60 UNITS

STIFFNESS OF 'CF' AS A SIMPLY SUPPORTED BEAM } = 3EI/L = 1/2

$$\therefore 104 = 60 + \theta FC/2$$

$$\therefore \theta FC = 88 \text{ UNITS} \quad \ldots\ldots\ldots\ldots \text{I}$$

*Figure 9*

*Hinge 1-2-b*—The rotation of hinge **1-2-b** will be solely due to the moments acting on members **FC** and **FI;** the moment acting on the vertical member **FE** does not have any influence on it due to the discontinuity at the hinge **2-2-a.** The rotation of **FC** can be estimated by creating artificial conditions at the hinge, i.e., at **F,** and then equating the artificial moment thus produced to the actual moment acting there, plus the product of the stiffness of **FC** (i.e., $3EI/L$) taken as a simply supported beam and the unknown rotation $\theta_{FC}$ to be evaluated. The product $(3EI/L)\theta$ or $(3EJ/h)\theta$ is always assumed to be positive for convenience. Figure 9 is self explanatory.

Similarly, the rotation of **FI** can be obtained as shown in Fig. 10, by repeating the procedure explained for member **FC.**

The rotation of **FC** is in the clockwise direction, but the rotation of **FI** is in the counterclockwise direction, thus tending to increase the angle of discontinuity. The rotation or the angle of discontinuity produced at hinge **1-2-b** is, therefore, the sum of the rotations obtained above.

*Hinge 2-2-c*—The angle of discontinuity at the hinge is produced by the rotations of the column **EF** and the beam **EH.** The moments acting on **EB** as well as **ED** have a tendency to rotate the hinges **1-1-b** and **2-1-a** respectively, but do not have any influence on the rotation of the hinge **2-2-c** because of discontinuity. The rotation of **EF** can be obtained as shown in Fig. 11 by creating artificial conditions, i.e., fixing end **E** and

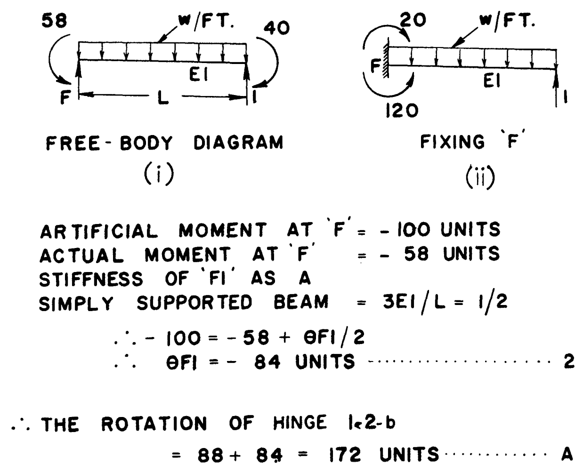



*Figure 10*

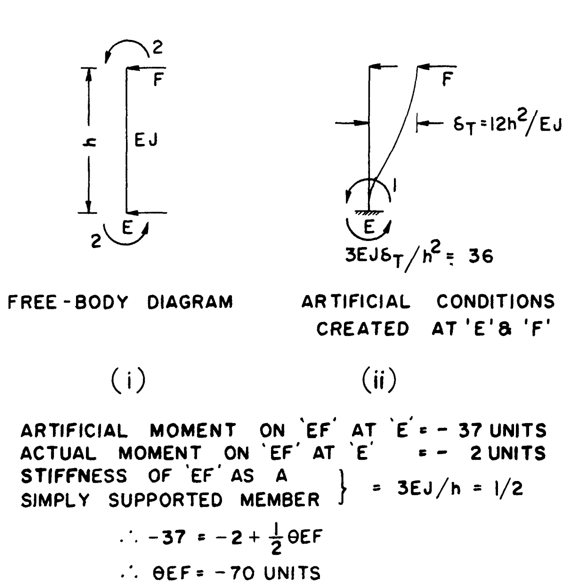



*Figure 11*

then moving end **F** laterally by an amount equal to the sway of the top story with respect to the bottom story. The artificial moment thus produced, when equated to the sum of the actual moment at **E** and the product $(3EJ/h)\theta_{EF}$, gives the value of $\theta_{EF}$. The rotation of beam **EH** can be obtained in a manner similar to that explained above for beam **FC.** See Fig. 12.

The net rotation or the angle of discontinuity is the difference between the rotations calculated above since they are in the same directions.

The rotations of all other hinges can be obtained in a similar manner and they are tabulated in Table 1. The magnitudes are the same as those obtained by Baker.[1]

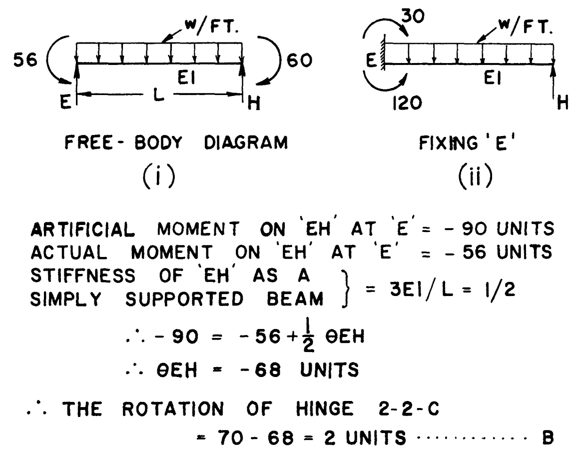



*Figure 12*

If the calculated rotations of the hinges are not well within the permissible value, or if they are not of the correct sign, either the location of the hinge or the moment acting at the hinge, or both, must be changed and the rotations recalculated. In the illustrative example only the first trial is presented.

## CONCLUSIONS

Since the concept of elastic moment distribution used in the method presented here is familiar to most engineers, the calculations of plastic hinge rotations become easier than with other existing methods. It can also be observed from the illustrative example explained above that the total number of operations involved to obtain the hinge rotations is much less than in Baker's method. This reduces considerably the time and the space required for the calculations, especially when the degree of indeterminacy of the structure is high.

## ACKNOWLEDGMENTS

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## NOMENCLATURE

| | | |
|---|---|---|
| $EI$ | = | Flexural rigidity of a beam |
| $EJ$ | = | Flexural rigidity of a column |
| $h$ | = | Height of a column |
| $L$ | = | Length of a beam |
| $M$ | = | Moment |
| $M_{p1}$ and $M_{p2}$ | = | Internal moments acting on the ends of a segment of the structure |
| $m$ | = | Maximum value of the external sway bending moment |
| $P$ | = | Horizontal load acting on each story |

$P_1$ and $P_2$ = Concentrated loads acting on the segment

$w$ = Intensity of load per foot run of the segment

$\theta$ = Rotation

$\phi$ = Curvature

$\delta$ = Relative displacement of either of the supports of a segment of the structure

$\delta_T$ = Horizontal sway of the top story relative to the bottom story

$\delta_b$ = Horizontal sway of the bottom story

$\theta_A$ = Net rotation of end **A**

$\theta_1$ = Rotation or the slope at the support of a segment due to the external loads only

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